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AN ARITHMETICAL DUAL OF KUMMER'S QUARTIC SURFACE.*

By E. T. Bell.

1. We shall use the well-established notations $\alpha \supset \beta$ to signify that α implies β, and $\alpha \equiv \beta$ to express the formal equivalence of α and β, viz. $\alpha \supset \beta$ *and* $\beta \supset \alpha$. Consider the homogeneous algebraic equation

$$\text{(A)} \qquad \Phi(x, y, z, w) = 0.$$

When x, y, z, w are the homogeneous coördinates of a point, (A) is the equation S' of a surface S. If x, y, z, w are assigned any other interpretation I, (A) is then called the equation E' of the equivalent E of S with respect to I. If $S' \equiv E'$, we shall call S, E duals of each other with respect to I; and if I is arithmetical, E is defined to be an arithmetical dual of S.

Let E be an arithmetical dual of S, and suppose that all the properties of integers implicit in E are transposed into a set P of properties of point configurations lying in a lattice space of the appropriate number of dimensions.† If now $P \supset E'$, then E, S being arithmetical duals, $P \supset S'$; and conversely $S' \supset P$. Hence since $P \equiv S'$ we may regard S in continuous point space as the equivalent or image of P' in the discrete lattice space. Another image of S will be glanced at in § 10. Clearly all of these considerations can be readily modified to hold for any system of equations representing loci in any space.

2. The most interesting of these duals appear to be those of the non-rational plane algebraic curves and the hyperelliptic r-folds in space of $r(r+1)/2$ dimensions. Obvious geometrical properties suggest analytical relations which dualize into interesting properties of numbers having many applications to the arithmetic of systems of higher forms. Thus we have a new application of geometry to numbers.

Here we briefly consider an arithmetical dual of the quartic surface with sixteen nodes, carrying the development up to the point of providing an immediate means for translating all the geometry of the surface directly

* Read before the San Francisco Section of the American Math. Soc., June 17, 1920.

† The E of Kummer's S considered in this paper can be interpreted in terms of lattice configurations lying upon two concentric spheres in a space of sixteen dimensions, but this is not developed here. This interpretation will be sufficiently evident from § 7 on applying the isomorphism there established to the rationalized form of the equation of S as given in Borchardt, *Crelle*, 83 (1877), p. 238; or Krause, *Die Transf. Hyperell. Funkt. erster Ord.* (Leipsig, 1886), p. 39; or Hudson, *Kummer's Quartic Surface* (Cambridge, 1905), p. 81. Rohn's parametric representations of S, *Math. Annalen*, 15 (1874), p. 315, based upon the transformation of the second order, give two more arithmetical duals distinct from E.

into arithmetic, the translation being reversible, so that from it the geometry can be recovered. Thus, for example, the 16_6 configuration of nodes and tropes is equivalent to a remarkable interlacing of properties of sets of three integers constructed from the decompositions of a pair of arbitrary integers into two sums of four squares, and conversely these imply the existence of the configuration.

3. Henceforth all letters l, m, n denote integers $\gtreqless 0$ unless further specified; *the l's are always even and the m's odd.* Write*

$$[\sum_{j=1}^{r} n_j] \equiv [n_1, n_2, \cdots, n_r],$$

and let either of these symbols denote $1, i, -1, -i$ $(i \equiv \sqrt{-1})$ according as

$$\sum_{j=1}^{r} n_j \equiv 0, 1, 2, 3 \bmod 4.$$

From the definition we have

$$[n_1, n_2, \cdots, n_r] = \prod_{j=1}^{r} [n_j]; \qquad [l] = (-1)^{l/2}; \qquad [m] = i(-1)^{(m-1)/2},$$

so that the symbol represents a real or imaginary unit according as the number of odd n_j is even or odd. We define $F(x, y, z)$ to be an arithmetical function if it exists and is single valued for x, y, z simultaneously integers $\gtreqless 0$, and otherwise is wholly arbitrary. If further conditions are imposed upon F it is called restricted; and unless the contrary is expressly noted, throughout the paper $F(x, y, z)$ denotes an unrestricted arithmetical function. An immediate consequence of this definition is of importance later (§ 10).

If

$$\sum_{j=1}^{r} F(n_{1j}, n_{2j}, n_{3j}) = \sum_{k=1}^{s} F(n_{1k}', n_{2k}', n_{3k}'),$$

then $r = s$, and in some order the triads (n_{1j}, n_{2j}, n_{3j}) are identical with the triads $(n_{1k}', n_{2k}', n_{3k}')$, two such being identical when and only when† $n_{ij} = n_{ik}'$ $(i = 1, 2, 3)$.

4. Let g_1, g_2, h_1, h_2 denote constant integers $\geqq 0$, and write $2n_j + g_j \equiv \nu_j$ $(j = 1, 2)$. Then the *general f-function* is defined by

$$f(\nu_1, \nu_2) \equiv f\begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(n_1, n_2)} = [h_1\nu_1, h_2\nu_2] F(\nu_1, \nu_2, \nu_1\nu_2).$$

* The $\equiv$ here means algebraic identity. There can be no confusion between this and the sign of formal equivalence, as in each case the context indicates which is meant. The [ ] notation is merely to avoid the printing of complicated exponents.

† If a formal proof of this proposition is desired, it can easily be constructed from section II of "Arithmetical Paraphrases," to appear shortly in the *Trans. Am. Math. Soc.*

he symbol $\begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}$ is the *mark* of this f, and is even or odd according as $g_1h_1 + g_2h_2$ is even or odd. According as its mark is even or odd, f is *quasi-even* or *quasi-odd*. If the f just written is quasi-even, the function

$$\varphi(\nu_1, \nu_2) \equiv \varphi \begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(n_1, n_2)} = [h_1\nu_1, h_2\nu_2]F(0, 0, \nu_1\nu_2)$$

is the *quasi-constant* corresponding to f. The integers ν_1, ν_2 are the *parameters of f or φ.*

For $g_1, g_2, h_1, h_2 = 0, 1$ there are thus 16 marks, of which 10 are even and 6 odd. Corresponding to these marks are 16 distinct f-functions which fall into sets of 4 each according to the residues mod 2 of the variables ν_1, ν_2:

$$(\nu_1, \nu_2) = (l_1, l_2),\ (m_1, l_2),\ (l_1, m_2),\ (m_1, m_2).$$

Transposing the entire theory of double theta marks to the present subject, we name each reduced mark, excluding (00), by the particular dyad of odd marks to which it is congruent.* If now (ij) is any one of the 16 reduced marks, (00) included, the $f(\nu_1, \nu_2)$ whose mark is (ij) is denoted by $f_{ij}(\nu_1, \nu_2)$, or simply by f_{ij}; and if (ij) is even, the corresponding quasi-constant is $\varphi_{ij}(\nu_1, \nu_2)$ or φ_{ij}. In this notation the order of i, j in the suffix is indifferent. We thus have the following system of 16 f-functions in which the parity of the variables is indicated by the l, m notation (§ 3):

$$\begin{array}{llll}
f_{00} = F(l_1, l_2, l_1l_2), & f_{23} = [l_1]f_{00}, & f_{45} = [l_2]f_{00}, & f_{16} = [l_1, l_2]f_{00}; \\
f_{12} = F(m_1, l_2, m_1l_2), & f_{13} = [m_1]f_{12}, & f_{36} = [l_2]f_{12}, & f_{26} = [m_1, l_2]f_{12}; \\
f_{56} = F(l_1, m_2, l_1m_2), & f_{14} = [l_1]f_{56}, & f_{46} = [m_2]f_{56}, & f_{15} = [l_1, m_2]f_{56}; \\
f_{34} = F(m_1, m_2, m_1m_2), & f_{24} = [m_1]f_{34}, & f_{35} = [m_2]f_{34}, & f_{25} = [m_1, m_2]f_{34}.
\end{array}$$

The six quasi-odd functions are

$$f_{13},\ f_{24},\ f_{46},\ f_{35},\ f_{26},\ f_{15};$$

and the ten quasi-constants are

$$\varphi_{00},\ \varphi_{12},\ \varphi_{56},\ \varphi_{34},\ \varphi_{23},\ \varphi_{14},\ \varphi_{45},\ \varphi_{36},\ \varphi_{16},\ \varphi_{25}.$$

Henceforth in all functions of f's and φ's, the φ's are regarded as being of degree zero, so that *in determining the degree of any expression involving f's and φ's the φ's are to be considered as absolute constants.* Thus $\varphi_{12}^2f_{34}^2 + \varphi_{35}^2f_{15}^2 = 0$ is a homogeneous linear relation between squares of f-functions.

5. The arithmetic-geometric translations mentioned in § 2 depend upon

* See Harkness and Morley, "Treatise on the Theory of Functions," ch. 8, whose notation we shall follow throughout: also Weber, "Über die Kummersche Fläche," u.s w., *Crelle*, 84 (1878), especially pp. 332–334.

a form of symbolic product, or henceforth simply product, of $(r - s)$ f-functions and s quasi-constants, $0 \leqq s \leqq r$. These products are taken with respect to a pair of integers n_1, $n_2 > 0$ of preassigned linear forms mod 4:

$$n_j \equiv t_j \bmod 4; \; t_j \text{ constant}, \quad (j = 1, 2).$$

Let all the g, h denote constants, each of which is one of the numbers 0, 1. Write

$$\begin{bmatrix} g_{1i}, & g_{2i} \\ h_{1i}, & h_{2i} \end{bmatrix} \equiv (i), \qquad 2n_{ji} + g_{ji} \equiv \nu_{ji} \quad (j = 1, 2),$$

and suppose that precisely t_1 of the g_{1i} each $= 1$, so that precisely t_1 of the ν_{1i} are odd, and similarly for t_2 and the g_{2i}, ν_{2i}. The f's or φ's for the marks (i), (k) are

$$f_i(\nu_{1i}, \nu_{2i}) \equiv f_i = [h_{1i}\nu_{1i}, h_{2i}\nu_{2i}]F(\nu_{1i}, \nu_{2i}, \nu_{1i}\nu_{2i}),$$
$$\varphi_k(\nu_{1k}, \nu_{2k}) \equiv \varphi_k = [h_{1k}\nu_{1k}, h_{2k}\nu_{2k}]F(0, 0, \nu_{1k}\nu_{2k}),$$
$$(i = s + 1, s + 2, \cdots, r; \; k = 1, 2, \cdots, s).$$

Put

$$h \equiv \sum_{i=1}^{r} (h_{1i}\nu_{1i} + h_{2i}\nu_{2i}), \qquad \xi_{12} \equiv \sum_{i=1}^{r} \nu_{1i}\nu_{2i}, \qquad \xi_j \equiv \sum_{j=s+1}^{r} \nu_{ji} \quad (j = 1, 2).$$

Then, the Σ extending to all $\nu_{ji} \gtreqless 0$ such that

$$n_j = \nu_{j1}^2 + \nu_{j2}^2 + \cdots + \nu_{jr}^2 \quad (j = 1, 2),$$

the *product with respect to* n_1, n_2 of the s quasi-constants and $(r - s)$ f-functions just written is defined by

$$\prod_{i=1}^{s}{}' \varphi_i(\nu_{1i}, \nu_{2i}) \cdot \prod_{i=s+1}^{r}{}' f_i(\nu_{1i}, \nu_{2i}) \equiv \Sigma[h]F(\xi_1, \xi_2, \xi_{12}),$$

and the *type** of this product is $\{t_1, t_2\}$, $\equiv \{t_2, t_1\}$. The accent in Π' indicates that the multiplication is purely symbolic.

Note that the sum on the right consists of only a finite number of terms. For if $N_r(n, s)$ is the total number of representations of n as a sum of r squares precisely s of which are odd and occupy fixed positions, the number of terms is $N_r(n_1, t_1)N_r(n_2, t_2)$. Note also (cf. § 3) that if $t_1 + t_2$ is even, $[h]$ is real. In the products connected with Kummer's surface it will be seen at once that each f_i, φ_k occurs an even number of times. Hence in these products $[h]$ is always real, and we need not again refer to this.

When a_j of the marks (1), (2), $\cdots$, (s) each $= j$, and b_k of the marks $(s + 1)$, $(s + 2)$, $\cdots$, (r) each $= k$, the product is written $\varphi_1^{a_1}\varphi_2^{a_2} \cdots f_1^{b_1}f_2^{b_2} \cdots$; and clearly the order of the factors $\varphi_1^{a_1}, \cdots, f_1^{b_1}, \cdots$ in this is im-

* The type is not important for this paper, and is included here merely for completeness. It plays an essential part in the detailed discussion of the arithmetical nodes and tropes on E, cf. § 9, second footnote.

material. Hence such multiplication of φ's and f's is commutative. We shall not be concerned here with its associative and distributive aspects. By definition $\varphi_i^0, f_i^0 = 1$; and unity in this multiplication is to have all the formal properties of unity in ordinary multiplication, so that unit factors may be suppressed.

The *parameters of the product* are n_1, n_2. These ultimately play a part, in one geometrical interpretation of the arithmetic, analogous to that of v_1, v_2, the parameters of a point on the Kummer surface; cf. § 9. Any product p_i being a sum with respect to given parameters, the meaning of $\Sigma_i a_i p_i$ where the a_i are absolute constants is evident. Note that

$$a_i p_i + a_j p_i = (a_i + a_j) p_i$$

when and only when both p_i are with respect to the same parameters. To indicate that each p_i in the sum is with respect to n_1, n_2 we write $n_1, n_2 \,|\, \Sigma_i a_i p_i$. If now

$$n_1, n_2 \,|\, \sum_i a_i p_i = 0, \tag{1}$$

it follows at once from the definitions that we may replace n_j by n_j' where $n_j' \equiv t_j \bmod 4$ $(j = 1, 2)$. For each pair (n_1', n_2'), (1) gives a relation

$$n_1', n_2' \,|\, \sum_i a_i p_i = 0 \tag{2}$$

between products. To signify that we are considering the totality of relations of the form (2) for all (n_1', n_2'), we shall write (1) without the $n_1, n_2 \,|$, thus,

$$\Sigma a_i p_i = 0. \tag{3}$$

It is important to note that (3) is of the form $P(f, \varphi) = 0$, where $P(f, \varphi)$ is a polynomial in f's and φ's.

Consider now $R(f, \varphi) = 0$ where $R(f, \varphi)$ is a rational algebraic function of f's and φ's. As yet this relation has no significance. By purely formal algebraic reductions $R(f, \varphi) = 0$ may be written $P(f, \varphi) = 0$, $P(f, \varphi)$ as above, and we define this to be the meaning of $R(f, \varphi) = 0$, again emphasizing the significance of the omission of $n_1, n_2 \,|$. Thus in all that follows, $R(f, \varphi) = 0$ and $P(f, \varphi) = 0$, its polynomial equivalent, are regarded as identical.

6. Denote by $R(\theta, c) = 0$ a rational algebraic relation with rational coefficients $\gtreqless 0$ between the sixteen double theta functions θ_{ij} and the related constants* c_{ij}.

* The notation is that of Harkness and Morley, loc. cit. The restriction that R be rational is merely to shorten the following proof. By a few simple changes the theorem may be restated for R algebraic with algebraic number coefficients, but this is not required for the dual of Kummer's surface.

By algebraic reductions $R(\theta, c) = 0$ may be cast in the form $P(\theta, c) = 0$, where now P is a polynomial with integral coefficients $\gtreqless 0$. We shall regard $R(\theta, c) = 0$, $P(\theta, c) = 0$ as identical statements. In these replace θ_{ij}, c_{ij} by f_{ij}, φ_{ij} respectively, and denote the results by $R(f; \varphi) = 0$, $P(f, \varphi) = 0$. It is not immediately evident, of course, that $P(f, \varphi) = 0$, or its formal equivalent $R(f, \varphi) = 0$, is true. The fundamental theorem is:

$$\{P(\theta, c) = 0\} \equiv \{P(f, \varphi) = 0\},$$

(the sign $\equiv$ being, as in the next section, that defined in § 1), or what is the same thing,

$$\{R(\theta, c) = 0\} \equiv \{R(f, \varphi) = 0\}.$$

No doubt it is easy to prove this from first principles. It is less tedious, however, to proceed as follows.

7. Clearly $P(f, \varphi) = 0$ is a finite sum relation of the form

$$\sum_k a_k F(\alpha_k, \beta_k, \gamma_k) = 0,$$

in which $a_k, \alpha_k, \beta_k, \gamma_k$ are integers and it is easily seen that if $g(x, y, z)$ is any restricted arithmetical function such that $g(x, y, z) = -g(-x, -y, -z)$, then $\Sigma \pm g(\pm x, \pm y, \pm x \times \pm y)$ is not identically zero. In each f, φ replace $F(\alpha_k, \beta_k, \gamma_k)$ by $\sin(\alpha_k x + \beta_k y + \gamma_k z)$, where x, y, z are parameters, and denote by $P_1(f, \varphi) = 0$ what $P(f, \varphi) = 0$ becomes under this substitution; and similarly for $F(\alpha_k, \beta_k, \gamma_k)$ replaced by $\cos(\alpha_k x + \beta_k y + \gamma_k z)$, giving $P_2(f, \varphi) = 0$. Write $P_3(f, \varphi) = iP_1(f, \varphi) + P_2(f, \varphi)$, $(i = \sqrt{-1})$.

We shall be concerned with the implications of the five propositions defined by

$\alpha \equiv \{P(f, \varphi) = 0\}$, $\beta \equiv \{R(\theta, c) = 0\}$, $\alpha_j \equiv \{P_j(f, \varphi) = 0\}$, $(j = 1, 2, 3)$, and will show that

$$(4)\quad \alpha \supset \alpha_3, \qquad (6)\quad \alpha_3 \supset \beta,$$
$$(5)\quad \beta \supset \alpha_3, \qquad (7)\quad \alpha_3 \supset \alpha;$$

whence from (4), (6) it will follow that $\alpha \supset \beta$, and from (5), (7) that $\beta \supset \alpha$, and therefore from these $\alpha \equiv \beta$, which is the theorem.

From the definition of F (§ 3), (4) is obvious. To prove (5) and (6) we observe that $P_3(f, \varphi)$ is the coefficient of $p_1^{n_1} p_2^{n_2}$ in $R(\theta, c) = 0$ when the expansion of the general theta function

$$\theta \begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(x/\pi,\ y/\pi,\ 4\tau_{11};\ 2z/\pi,\ 4\tau_{22})}, \quad (z = \pi\tau_{12}/2),$$

is written in the form

$$\Sigma p_1^{\nu_1^2} p_2^{\nu_2^2} [h_1\nu_1, h_2\nu_2] \exp i(\nu_1 x + \nu_2 y + \nu_1\nu_2 z) \quad (p_j = \exp i\pi\tau_{jj}),$$

the Σ referring to all $\nu_j = 2n_j + g_j$ $(j = 1, 2)$.

To prove (7) consider two restricted arithmetical functions (cf. § 3) ψ_1, ψ_2, the only restrictions upon them being

$$\psi_j(u, v, w) = (-1)^j \psi_j(-u, -v, -w), \quad (j = 1, 2),$$

and consider the propositions α_j' defined by the formal equivalences

$$\alpha_j' \equiv \{\sum_k a_k \psi_j(\alpha_k, \beta_k, \gamma_k) = 0\}, \quad (j = 1, 2).$$

Then it may be shown without difficulty* that

(8) $$\alpha_j \supset \alpha_j', \quad (j = 1, 2).$$

Now choose for the ψ_j the restricted arithmetical functions defined by

$$2\psi_j(u, v, w) = F(u, v, w) + (-1)^j F(-u, -v, -w), \quad (j = 1, 2),$$

which obviously satisfy the restrictions imposed upon the ψ_j. Substitute these values of the ψ_j in the α_j' respectively, getting α_j''. Then from (8)

(9) $$\alpha_j \supset \alpha_j'', \quad (j = 1, 2).$$

If now $i\alpha_1$ denotes the result of replacing $\sin(\alpha_k x + \beta_k y + \gamma_k z)$ in α_1 by i times itself, we have $i\alpha_1 \supset \alpha_1$, and hence from (9)

(10) $$i\alpha_1 \supset \alpha_1'' \quad \textit{and} \quad \alpha_2 \supset \alpha_2''.$$

But obviously

$$(\alpha_1'' \textit{ and } \alpha_2'') \supset \alpha, \text{ also } (i\alpha_1 \textit{ and } \alpha_2) \equiv \alpha_3;$$

while from (10),

$$(i\alpha_1 \textit{ and } \alpha_2) \supset (\alpha_1'' \textit{ and } \alpha_2'').$$

Hence $\alpha_3 \supset \alpha$, which is (7).

8. The theorem of § 6 enables us to transpose the entire theory of the rational algebraic relations between the θ_{ij}, c_{ij} to the present subject. In particular there is here a system of 13 Rosenhain hexads of which one is†

$$f_{13}, f_{35}, f_{15}, f_{24}, f_{46}, f_{26}$$

such that there is a linear relation (cf. § 4 end) between the squares of any four f-functions belonging to the same hexad; also the sixteen f_{ij} fall into 60 Göpel tetrads, the four f-functions in a tetrad being connected by a homogeneous biquadratic relation. Again, between the squares of any five f-functions, no four of which belong to a hexad, there is a linear relation; and we may state the general theorem* that the squares of four f-functions are linearly related when and only when their marks belong to a Rosenhain

* This is included as a very special case of a theorem proved in "Arithmetical Paraphrases," Part I, Sec. II, cited in § 3, footnote.

† The complete system may be written down from the table in Harkness and Morley, loc. cit., pp. 353–354.

‡ Harkness and Morley, loc. cit., p. 368

hexad, and the squares of any five f-functions no four of which are in one hexad are linearly related. All of these results have immediate arithmetical interpretations in terms of representations of n_1, n_2 either as sums of four squares (for the quadratic relations), or as sums of sixteen squares (for the biquadratic). The subject being extensive we shall not go into it here.

By the theorem of § 6, the Göpel relations for theta functions (and constants) are formally equivalent to the same for f-functions (and quasi-constants), so that the Kummer surface gives in this way 60 arithmetical duals. We shall sketch a means for translating the arithmetic into terms of configurations of lattice points in S_r (space of r dimensions), a lattice point being one all of whose coördinates are integers. Finally it will be shown that the lattice properties in S_r can be mapped onto a point configuration in S_3. In each case the configuration is formally equivalent to the arithmetical dual, which in turn is formally equivalent to the equation of the surface, so that the existence of the point configuration implies that of the surface, and conversely.

9. Let ν_a, ν_a' be the parameters (§ 3) of the f whose mark is a, and consider the tetrad $T \equiv (f_a, f_b, f_c, f_d)$. As the eight parameters $\nu_a, \nu_a', \cdots, \nu_d, \nu_d'$ take all their possible values, we shall say that T generates a spread. The totality of all spreads obtained by choosing a, b, c, d and F (cf. § 3) in all possible ways is called f-space, and each T for constant values of the eight parameters involved is a point in f-space (or the coördinates of a point in f-space). The analogously defined $(\varphi_a, \varphi_b, \varphi_c, \varphi_d)$, in which the φ's are quasi-constants, is a singular point of f-space. The coördinates of a general point in f-space are (f_a, f_b, f_c, f_d), in which all the parameters are general. In the same way we define the general f-line $(f_a, f_b, f_c, f_d, f_e, f_g)$, $a, \cdots, g$ being any six marks, and similarly a singular f-line on replacing f's by φ's when all the marks are even. We remark that, the p's being products, (p_1, p_2, p_3, p_4) is clearly an f-point; and likewise for hexads of products and f-lines. The analogies between f-points and lines and the points and lines of S_3 are obvious, and need not be further elaborated.

Suppose now that between the coördinates $(f_1, f_2, f_3, f_4) \equiv (x, y, z, w)$ of an f-point there is a relation of the form (A), § 1. Remembering that by § 4 (end) quasi-constants are to be considered as being of degree zero, and that by § 5(3) each product in (A) is with respect to the same parameters n_1, n_2 which are general, we shall call (A) the equation of the surface E (§ 1), in f-space, $x = f_1$, $y = f_2$, $z = f_3$, $w = f_4$ the parametric equations of E, and n_1, n_2 the parameters of a point on E. Replacing f-functions throughout in the above by quasi-constants, we similarly define singular surfaces in f-space. The singular surfaces have no immediate geometrical

analogues; their interest is arithmetical, and need not be considered here.* In the same way is defined the line-equation of an f-surface or of a singular f-surface. As one parameter varies, the other being constant, the associated point on the surface traces out what may be called an f-curve; and the analogy with geometry may obviously be continued step by step. Thus, to define a node on the f-surface E, we equate to zero the four formal partial derivatives with respect to the f-coördinates. It does not follow, of course, that any one of the four new relations thus derived between products is true. But if it be possible in the four derived relations so to replace the variables (f-functions) x, y, z, w by quasi-constants $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ respectively that the relations become simultaneously true, $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ is defined to be a node on E. It follows from § 7 that the E of Kummer's S has sixteen nodes, and it is sufficiently evident that the geometry of nodes on S can be transferred to a 'geometry' of 'nodes' on E, and conversely.† All of this geometry on E has a simple interpretation in S_3 which we shall briefly consider next, showing how the translation is effected in a general case. It is not the most general case, for we have excluded the possibility of imaginary $[h]$, but the perfectly general case presents no difficulty, and is treated in essentially the same way. In the case considered we end with the identity of two point configurations in a lattice space of three dimensions; in the most general case the final result is that two pairs of point configurations are thus severally identical. The case treated appears to be the more important. It includes the E of Kummer's S.

10. Consider three fixed points U, V_1, V_2 and two concentric spheres σ_1, σ_2, centers at the origin of the respective radii a_1, a_2 in S_r (§ 9), and let T_j denote the tangent plane (tangent S_{r-1}) at the point P_j on σ_j $(j = 1, 2)$. Let $\delta_j, \delta_{12}, \delta_{21}$ denote the respective perpendicular distances from U to T_j, from P_1 to T_2 and from P_2 to T_1. It is easily seen that

$$a_1\delta_{21} + a_1^2 = a_2\delta_{12} + a_2^2.$$

With reference to rectangular axes in S_3 the point P_{12} whose coördinates

* Interpreted similarly to the non-singular surfaces in § 10 (end), they give point configurations lying in one plane.

† The equivalent arithmetic is obtained in an obvious manner. For example the the theorem that a particular set of six nodes is in one plane is formally equivalent to a theorem that six products (§ 5), in which $r = s = 2$, are linearly related. The six products are sums of terms of the form $[h]F(\xi_1, \xi_2, \xi_{12})$, where the ξ's refer in this example to all the representations of the parameters n_1, n_2 in two sums of four squares, either 2 or 4 of which are odd and the rest even. The number of odd squares, also the $[h]$, are not the same for all sets of six coplanar nodes. Finally, as in the equivalent theta theory, all these relations between (symbolic) products can be summed up in the statement that a certain matrix is orthogonal, cf. Hudson, loc. cit., chs. III, XVI.

are either of the identical triads

$$(a_1\delta_1 + a_1^2,\ a_2\delta_2 + a_2^2,\ a_1\delta_{21} + a_1^2), \qquad (a_1\delta_1 + a_1^2,\ a_2\delta_2 + a_2^2,\ a_2\delta_{12} + a_2^2),$$

is called the image through U of the pair P_1, P_2, or when U is understood, simply the image of P_1, P_2.

Let δ_j' denote the perpendicular distance of V_j to T_j. Then it is readily seen that if the squares of the radii a_1, a_2 are integers, and V_j, P_j lattice points, h_j, defined by

$$h_j = a_j\delta_j' + a_j^2, \qquad h \equiv h_1 + h_2, \qquad (j = 1, 2),$$

are integers. We shall call h_j the index of P_j, and h the index of P_{12}. *Henceforth* P_1, P_2 *are lattice points.* Images P_{12} are now segregated into four classes K_j $(j = 0, 1, 2, 3)$, all images in K_j having their indices $\equiv j \bmod 4$. We shall consider henceforth only K_0 and K_2. A configuration C_0 of images in S_3 is called even when all its images belong to K_0; if all the images of C_2 belong to K_2, C_2 is odd. Several even C_0's together are regarded as forming a single C_0; likewise for C_2's, so that the C_j is the logical sum of all the images in the several C_j', C_j'', $\cdots$,

$$C_j = C_j' + C_j'' + \cdots \quad (j = 1, 2).$$

If in C_j a particular image P_{12} occurs precisely k times, P_{12} is *multiple of order* k in C_j; and if all the images of C_0 coincide with all those of C_2, C_0 and C_2 are identical when and only when images occupying the same positions in both are of equal multiplicities.

Returning to § 5 we shall consider only the case in which h there defined is even, so that $[h]$ is real. In the notation of § 5 choose for the radii of σ_1, σ_2 of this section, $\sqrt{n_1}$, $\sqrt{n_2}$, let the origin be the common center, and take for U, V_j, P_j the points

$$\begin{aligned} U &\equiv (0, 0, \cdots, 0, 1, 1, \cdots, 1), \quad (s \text{ zeros}, r - s \text{ units}),\\ V_j &\equiv (h_{j1}, h_{j2}, \cdots, h_{jr}), \quad (j = 1, 2),\\ P_j &\equiv (\nu_{j1}, \nu_{j2}, \cdots, \nu_{jr}), \quad (j = 1, 2). \end{aligned}$$

Assume $r \gtrless 4$ (the only cases of importance), so that, any integer > 0 being in several ways a sum of four integral squares, σ_j always passes through lattice points P_j. For the U, V_j, P_j as just defined, it is easily seen that P_{12}, the image through U of P_1, P_2, is, in the notation of § 5, (ξ_1, ξ_2, ξ_{12}), and that h as there defined is its index. Hence to each term $[h](\xi_1, \xi_2, \xi_{12})$ of the product in § 5 corresponds an image of odd or even index, and to all the terms in the product correspond an even and an odd configuration.

Suppose now that we have a homogeneous algebraic relation $H = 0$

of degree $(r - s)$ (cf. § 4 end) between k such products. The k even configurations corresponding respectively to the k products form a single C_0; likewise the k odd configurations form a single C_2, and by § 3 (end) we see at once that $H = 0$ is formally equivalent to the statement that these single odd and even configurations are identical.

It should be of interest, for physical reasons, to find the corresponding discrete image of the special case of Kummer's surface known as the wave surface. The arithmetic for this case appears to be less elegant than that for the general surface.

INCIDENCES OF STRAIGHT LINES AND PLANE ALGEBRAIC CURVES AND SURFACES GENERATED BY THEM.

By Arnold Emch.

1. Introduction.

According to Lüroth* the analytic method of investigating problems of this sort, as indicated by Clebsch,† on account of its complexity, is not practicable. Thus, even for the simple problems connected with incidences of straight lines and conics in ordinary space, Lüroth prefers to use a purely geometric method for their solution.

Some of these problems, involving merely the number of solutions in each case, have subsequently also been solved by the methods of enumerative geometry. But, without wishing to detract from the great value of this geometry, it must be said that the applications of its processes are purely mechanical and do not afford a clear insight into the geometric organism. For example, enumerative geometry does not teach how to get the solutions effectively, or constructively; it is satisfied with getting their number.

In this paper I shall show that the particular problems considered by Lüroth, and more general ones, can, after all, be solved by a relatively very simple analytical method.

2. Incidence of Point, Straight Line, Plane, and Surface.

Let $(a) \equiv (a_1, a_2, a_3, a_4)$ and $(b) \equiv (b_1, b_2, b_3, b_4)$, be two points determining a straight line l (hereafter we shall use simply the term *line*, meaning straight line). The homogeneous coördinates of this line are

$$\rho p_{ik} = a_i b_k - a_k b_i; \qquad i, k = 1, 2, 3, 4; \; i \neq k,$$

so that the p_{ik}'s satisfy the identity

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} \equiv 0.$$

The line l intersects a plane (α)

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0$$

* "Ueber die Anzahl der Kegelschnitte, welche acht Geraden im Raume schneiden," *Crelle's Journal*, Vol. 68, pp. 185–192 (1868).

† Ibid., Vol. 59, p. 1 ff.

in a point $(x) \equiv (x_1, x_2, x_3, x_4)$, whose coördinates are easily found as

$$
(1) \qquad \begin{aligned}
\rho x_1 &= \quad 0 \quad + \alpha_2 p_{12} + \alpha_3 p_{13} + \alpha_4 p_{14}, \\
\rho x_2 &= - \alpha_1 p_{12} + \quad 0 \quad + \alpha_3 p_{23} + \alpha_4 p_{24}, \\
\rho x_3 &= - \alpha_1 p_{13} - \alpha_2 p_{23} + \quad 0 \quad + \alpha_4 p_{34}, \\
\rho x_4 &= - \alpha_1 p_{14} - \alpha_2 p_{24} - \alpha_3 p_{34} + \quad 0 \quad .
\end{aligned}
$$

Incidentally, it is noticed that this represents a singular null-system, whose determinant is the square of the lefthand side of the identity satisfied by the coördinates of the line, and in which to every plane (α) not passing through l corresponds its point of intersection with l.

If we now substitute (1) in the point equation of a surface S_n of order n

$$(2) \qquad f(x_1, x_2, x_3, x_4) = 0,$$

we obtain an equation

$$(3) \qquad F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0,$$

which is homogeneous and of order n in the α's, and also in the p_{ik}'s, and of order 1 in the coefficients of (2). The values of (1) which satisfy (2) are evidently the coördinates of a point on the surface S_n, but, being in the form (1), they are evidently also the coördinates of a point on l and (α), and, consequently, the coördinates of a point of intersection of l and S_n. Any plane (α) which passes through a point of intersection of l and S_n, will therefore satisfy (3). Conversely, every solution (β) of (3) represents a plane through one of the points of intersection of l and S_n. This follows immediately from the identity

$$F(\beta_1, \beta_2, \beta_3, \beta_4) \equiv f(\beta_2 p_{12} + \beta_3 p_{13} + \beta_4 p_{14}, \cdots),$$

in which the point $\rho x_1 = 0 + \beta_2 p_{12} + \beta_3 p_{13} + \beta_4 p_{14}, \cdots$ evidently lies on S_n, (β) and l. From this follows that the lefthand side of (3) resolves into n linear factors in the α's, which, set equal to zero, represent the equations of the intersections of l and S_n, or the bundles of planes through these points. (3) is therefore a necessary and sufficient condition for the incidence of point, line, plane, and surface.

When a plane (α) passes through l then its coördinates expressed by the coördinates of l, and a parameter λ are

$$
(4) \qquad \begin{aligned}
\rho\alpha_1 &= - p_{34}, \\
\rho\alpha_2 &= \lambda p_{34}, \\
\rho\alpha_3 &= \lambda p_{42} + p_{14}, \\
\rho\alpha_4 &= \lambda p_{23} - p_{13}.
\end{aligned}
$$

For these values of the α's, the coördinates of (x) given by (1) vanish identically, i.e., the point (x) as the intersection of (α) and l is indetermined.

As (2) is of degree n, and as each x in (2) vanishes when the α's have the values (4), it is evident that for every set of values (4), determined by λ, (2) vanishes to the nth order; in other words, *every plane, determined by λ in (4), is an n-fold plane of the degenerate surface (Σ) of class n.* This is also geometrically evident. (3) consists of the n bundles of planes through the n points of intersection of l with S_n, and a plane through l is common to all n bundles, and is therefore n-fold in the totality of bundles as given by (3).

3. Application to Plane n-ics in Space Cutting a Fixed Independent Plane n-ic in n Points. The Surface E.

Considering general curves and surfaces, if not stated otherwise, a surface S_n of order n which passes through a fixed plane n-ic C_n^0 depends on

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n+3)}{2}$$

parameters. Let C_n be another plane n-ic cutting each of $[n(n+3)/2] - n$ lines l, and the C_n^0 in n collinear points. If S_n is also made to contain C_n, the number of independent parameters left for the complete determination of S_n is

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n+3)}{2} - \frac{n(n+3)}{2} + n = \frac{(n-1)n(n+1)}{6}.$$

Choosing this as the number of fixed points P through which S_n shall pass, imposes upon S_n (containing C_n^0)

$$\frac{n(n+3)}{2} + \frac{(n-1)n(n+1)}{6} = \frac{n(n^2+3n+8)}{6}$$

given conditions, so that there are

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n^2+3n+8)}{6} = \frac{n(n+1)}{2}$$

independent parameters left. Hence there are $[n(n+1)/2]+1$ linearly independent surfaces $\phi_1, \phi_2, \phi_3, \cdots, \phi_{[n(n+1)/2]+1}$ through C_n^0 and the $n(n^2-1)/6$ fixed points P, and the general S_n may be written in the form

$$(5) \qquad a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \cdots + a_{[n(n+1)/2]+1} \cdot \phi_{[n(n+1)/2]+1} = 0.$$

Every plane (α) cuts the C_n^0 and the $n(n+1)/2$ lines l in points of a plane n-ic C_n which together with C_n^0 and the fixed points P determine an S_n uniquely. But when (α) passes through one of the points P, the corresponding S_n degenerates into the plane (α) itself, and an $(n-1)$-ic.

Now let l_m be one of $[n(n+1)/2]+1$ independent lines l, and denote its

homogeneous coördinates by p_{ik}^m. Substituting these values in (1) gives the coördinates of the point of intersection (x) of the line l_m with the plane (α). Let $\phi_1^m, \phi_2^m, \phi_3^m, \cdots$ be the result of the substitution of the coördinates of (x) in $\phi_1, \phi_2, \phi_3, \cdots$; then

$$(6) \qquad a_1\phi_1^m + a_2\phi_2^m + a_3\phi_3^m + \cdots + a_{[n(n+1)/2]+1}\phi^m_{[n(n+1)/2]+1} = 0$$

is the condition that the point of intersection of l_m and (α) lies on S_n. Repeating this process for all lines l_m, $m = 1, 2, 3, \cdots, [n(n+1)/2] + 1$; and the same (α), leads to $[n(n+1)/2] + 1$ equations of order n in the α's, whose consistency is assured when the determinantal equation

$$(7) \qquad \begin{vmatrix} \phi_1^1 & \phi_2^1 & \phi_3^1 & \cdots & \phi^1_{[n(n+1)/2]+1} \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi^2_{[n(n+1)/2]+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \phi_1^{n(n+1)/2+1} & \phi_2^{n(n+1)/2+1} & \phi_3^{[n(n+1)/2]+1} & \cdots & \phi^{[n(n+1)/2]+1}_{[n(n+1)/2]+1} \end{vmatrix} = 0$$

is satisfied. This is an equation of order

$$n\left(\frac{n(n+1)}{2} + 1\right) = \frac{n^3 + n^2 + 2n}{2}$$

in the α's. But the bundles of planes through the fixed points P, determining degenerate S_n's must be subtracted, so that the degree of the equation proper is

$$\frac{n^3 + n^2 + 2n}{2} - \frac{n(n^2 - 1)}{6} = \frac{n(2n^2 + 3n + 7)}{6}.$$

Every plane (α) of the reduced equation (7) cuts the C_n^0 in n points and the $[n(n+1)/2] + 1$ lines l in points which lie on a plane n-ic C_n. Hence

THEOREM 1. *The planes, each of which cuts singly each of $[n(n+1)/2] + 1$ lines l and an independent plane n-ic C_n^0 together in a system of points which lies on a plane n-ic C_n, form a surface E of class*

$$\frac{n(2n^2 + 3n + 7)}{6}.$$

4. DEVELOPABLE SURFACE D WHOSE PLANES CUT A FIXED PLANE n-IC AND $[n(n+1)/2] + 2$ INDEPENDENT LINES IN POINTS OF PLANE n-ICS.

Let E_1 be the surface determined by the lines

$$l_1, l_2, l_3, \cdots, l_{n(n+1)/2}, l_{[n(n+1)/2]+1};$$

E_2 the surface determined by the lines

$$l_1, l_2, l_3, \cdots, l_{n(n+1)/2}, l_{[n(n+1)/2]+2},$$

according to theorem (1); E_1 and E_2 have a developable surface D^* of class

$$\left[\frac{n(2n^2+3n+7)}{6}\right]^2$$

in common. But through each of the $n(n+1)/2$ lines common to both E_1 and E_2 there is a pencil of planes, whose planes cut the remaining lines and C_n^0 in points which lie on an n-ic for each such plane. These planes are each n-fold in E_1 and E_2, and as they all contain n-ics which cut one of the lines l improperly, each pencil of planes through the lines common to E_1 and E_2 must be deducted n^2-fold from D^*. This leaves for the class of the developable surface D proper the number

$$\left[\frac{n(2n^2+3n+7)}{6}\right]^2 - n^2\frac{n(n+1)}{2} = \frac{n^2(4n^4+12n^3+19n^2+24n+49)}{36}.$$

Hence

THEOREM 2. *The planes, each of which cuts singly each of* $[n(n+1)/2]+2$ *lines* l *and an independent plane* n*-ic* C_n^0 *together in a system of points which lies on a plane* n*-ic* C_n, *form a surface* D *of class*

$$\frac{n^2(4n^4+12n^3+19n^2+24n+49)}{36}.$$

5. NUMBER OF PLANE n-ICS CUTTING SINGLY EACH OF $[n(n+1)/2]+3$ LINES l AND A FIXED PLANE n-IC IN n POINTS.

The surface D determined by the lines

$$l_1,\ l_2,\ l_3,\ \cdots,\ l_{n(n+1)/2},\ l_{[n(n+1)/2]+1},\ l_{[n(n+1)/2]+2},$$

according to theorem (2), and the surface E determined by the lines

$$l_1,\ l_2,\ l_3,\ \cdots,\ l_{n(n+1)/2},\ l_{[n(n+1)/2]+3},$$

according to theorem (1), have

$$\frac{n^2(4n^4+12n^3+19n^2+24n+49)}{36}\cdot\frac{n(2n^2+3n+7)}{6}$$

common solutions, including the improper ones. To get the number of the latter we must take into account that there are $n(2n^2+3n+7)/6$ planes through each of the $n(n+1)/2$ lines common to D and E, of which each cuts the remaining $[n(n+1)]/2+2$ lines of a plane n-ic belonging to both D and E. Each of these planes is a multiple plane of order n in both surfaces. The number of proper planes cutting singly all $[n(n+1)/2]+3$ lines in points on plane n-ics is therefore

$$\frac{n^2(4n^4+12n^3+19n^2+24n+49)}{36}\cdot\frac{n(2n^2+3n-7)}{6}$$

$$-\frac{n(n+1)}{2}\cdot\frac{n(2n^2+3n+7)}{6}\cdot n^2$$

$$=\frac{n^2(8n^6+36n^5+66n^4+99n^3+123n^2+89n+343)}{216}.$$

Denoting this number by N, the result may be stated in

THEOREM 3. *There are N plane n-ics which cut singly each of $[n(n+1)/2]+3$ lines and which cut, in each case, a fixed plane n-ic in n points.*

6. A CERTAIN SURFACE GENERATED BY PLANE n-ICS.

Given a fixed plane n-ic C_n^0 and $n(n+1)/2$ independent lines l. Consider the pencil of planes through another independent fixed line s. Every plane of this pencil cuts C_n^0 and the l's in points on an n-ic C_n. As the plane describes the pencil through s, the corresponding plane n-ic C_n describes a certain surface, whose order is determined by the number of points in which any other independent line, say $l_{[n(n+1)/2]+1}$ cuts the surface. Now we know the class of the surface E determined by $l_1, l_2, l_3, \cdots, l_{n(n+1)/2}, l_{[n(n+1)/2]+1}$. The number of plane n-ics through s cutting all these $[n(n+1)/2]+1$ lines l is, of course, equal to the class of E. Hence $l_{n(n+1)/2]+1}$ cuts the surface in a number of points equal to the class of E, and we have

THEOREM 4. *The plane n-ics, whose planes form a pencil, and which cut a fixed plane n-ic in n points, and which cut singly each of $n(n+1)/2$ lines, generate a surface of order*

$$\frac{n(2n^2+3n+7)}{6},$$

in which the axis s of the pencil is an

$$\left[\frac{n(2n^2+3n+7)}{6}-n\right]=\frac{n(2n^2+3n+1)}{6}$$

fold line.

The last part of this theorem is obvious, since a plane through s cuts the surface, outside of s, in a plane n-ic only.

7. EXAMPLE OF CIRCLES.

Let the surfaces S_n be spheres, so that the fixed plane n-ic C_n^0 is the imaginary sphero-circle (absolute) at infinity. In this case (as well as for any other conic as C_n^0) $[n(n+1)/2]+3=6$, $N=175$, and theorem 3 assumes the form of a corollary:

There are 175 circles cutting singly each of 6 independent lines.

8. GENERALIZATION OF LÜROTH'S PROBLEM.

In the paper referred to in the introduction Lüroth investigates incidences of conics and lines and certain surfaces enveloped by the planes of such conics, or generated by such conics. His concluding theorem states that there are 92 conics which cut singly each of 8 independent lines in space. I shall generalize this result by establishing the number of plane n-ics which cut singly each of $[n(n+3)/2]+3$ independent lines in space.

The key for the solution of this problem is again furnished by the establishment of the class of the surface E formed by the planes of the n-ics cutting singly each of $[n(n+3)/2]+1$ lines l. For this purpose I shall follow briefly a line of reasoning entirely similar to that for the surface E in § 3. Accordingly it is found that there are $(n^2+n+2)/2$ linearly independent surfaces $\phi_1, \phi_2, \phi_3, \cdots$ of order n which pass through n definite fixed lines l and $(n^3-3n^2+2n)/6$ independent fixed points P. Hence any surface S_n of order n through these fixed elements may be represented by a linear polynomial

$$S_n = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \cdots$$

in the ϕ's. The condition that a plane (α) shall cut $(n^2+n+2)/2$ other independent lines l in points which lie on an S_n is equivalent with the same number of equations of the form

$$a_1\phi_1{}^m + a_2\phi_2{}^m + a_3\phi_3{}^m + \cdots = 0.$$

This leads to a determinantal equation of order $n(n^2+n+2)/2$ in the α's. But, again, the bundles of planes through the points P must be deducted, so that a reduced equation of degree

$$\frac{n(n^2+n+2)}{2} - \frac{n^3-3n^2+2n}{6} = \frac{n^3+3n^2+2n}{6}$$

in the α's is left. Thus, in analogy with theorem (1) we have

THEOREM 5. *The planes each of which cuts singly each of* $[n(n+3)/2]+1$ *independent fixed lines in space in points which lie on a plane n-ic is a surface E of class*

$$\frac{n^3+3n^2+2n}{3}.$$

By Lüroth's method, extended to the case of n-ics, and in analogy with the procedure in §§ 4 and 5, it is found that the class of the developable surface D, formed by the planes which cut $[n(n+3)/2]+2$ lines l in points on n-ics, is

$$\frac{n^2(2n^4+12n^3+17n^2-3n+8)}{18}.$$

In case of conics, $n = 2$, this number, as shown by Lüroth, has to be reduced by 10, so that for conics the class of D is 34.

Finally the number of plane n-ics cutting $[n(n+3)/2] + 3$ independent lines in space is

$$\frac{n^3(n^2+3n+2)(n^4+6n^3+4n^2-15n+4)}{27}.$$

For $n = 2$ this number is reduced by 20 (Lüroth, loc. cit.) so that the number of conics cutting singly each of 8 lines is 92.

ON THE THEOREMS OF GAUSS AND GREEN.

By Vincent C. Poor.

Part I.

The Divergence of a Vector.

Definitions.—Among the definitions for the divergence of a vector, we have the following:

$$\operatorname{div} \mathbf{u} = \underset{\tau \to 0}{\text{Limit}} \frac{\int \mathbf{u} \times \mathbf{n}\, d\sigma}{\tau},$$

where $\mathbf{u}$ is a continuous vector point function whose scalar product $\mathbf{u} \times \mathbf{n}$ with the outward unit normal $\mathbf{n}$ of the surface σ is integrable over the boundary of the region τ, or over the bounding surface of any sub-region of the region τ. This definition may be looked upon as a differential form of Gauss's Theorem:

$$(1) \qquad \int \operatorname{div} \mathbf{u}\, d\tau = \int \mathbf{u} \times \mathbf{n}\, d\sigma.$$

In the elements of vector analysis, Burali-Forti and Marcolongo give the following definition:

$$\operatorname{div} \mathbf{u} = \{\operatorname{grad} (\mathbf{u} \times \mathbf{a}) - \operatorname{rot} (\mathbf{u} \wedge \mathbf{a})\} \times \mathbf{a},^*$$

where $\mathbf{a}$ is an arbitrary constant vector. The symbol, $\wedge$, read vec, is Burali-Forti and Marcolongo's symbol for the vector product of two vectors, and the abbreviations, grad and rot, are the gradient, and the rotation or curl of a vector respectively.

The former definition is not convenient for proving directly such theorems as

$$\begin{aligned} \operatorname{div} m\mathbf{u} &= m \operatorname{div} \mathbf{u} + \operatorname{grad} m \times \mathbf{u}, \\ \operatorname{div} \mathbf{u} \wedge \mathbf{v} &= \mathbf{u} \times \operatorname{rot} \mathbf{v} - \mathbf{v} \times \operatorname{rot} \mathbf{u} \end{aligned}$$

and the uniqueness and existence of the limit might be difficult to establish without resorting to a coördinate system. The second definition seems very artificial, to say the least, and not of a form to be readily remembered.

Some of these objections are obviated through the following definition, which, expressed in the notation of Burali-Forti and Marcolongo, reads as follows:

$$(2) \quad \operatorname{div} \mathbf{u} \cdot dP \times \delta P \wedge \vartheta P = d\mathbf{u} \times \delta P \wedge \vartheta P + \delta \mathbf{u} \times \vartheta P \wedge dP + \vartheta \mathbf{u} \times dP \wedge \delta P,$$

where the point differentials dP, δP, ϑP are any three non-complanar vec-

* "Elements de Calcul Vectoriel, Burali-Forti et Marcolongo," p. 71, article 3.

tors. This form is both natural and convenient; natural, since it may be easily seen to be a differential form of Gauss' Theorem, and convenient because of the differential operators involved. The expanded forms of div $m\mathbf{u}$ and div $\mathbf{u} \wedge \mathbf{v}$ are readily deducible. By certain changes this form of definition may be deduced from Burali-Forti's definition* for the first invariant of a particular homography.

Gauss' Theorem.—Also from this definition the integral form of Gauss' Theorem may be established. To show this, the region τ, bounded by the surface σ, may be divided into cells by passing planes parallel to the planes determined by the arbitrary vectors dP, δP, ϑP, taken in pairs. The right member of (2) is the resultant flux of the vector $\mathbf{u}$, through the surface of each cell. The resultant flux of $\mathbf{u}$ through each face common to two adjacent cells is zero, since the outward normals are of opposite sense. Upon applying the fundamental law of the integral calculus the right member in the limit is seen to be the flux integral of the vector $\mathbf{u}$ through the surface σ, and Gauss' Theorem follows.

Uniqueness.—To prove the uniqueness of the idea defined by (2) we can, in the usual way, assume another operator operating on $\mathbf{u}$, div′ $\mathbf{u}$. Then by definition

$$\text{(3)}\quad \operatorname{div}'\mathbf{u}\cdot dP\times\delta P\wedge\vartheta P = d\mathbf{u}\times\delta P\wedge\vartheta P+\delta\mathbf{u}\times\vartheta P\wedge dP+\vartheta\mathbf{u}\times dP\wedge\delta P.$$

Subtracting (3) from (2) we have

$$(\operatorname{div}\mathbf{u}-\operatorname{div}'\mathbf{u})dP\times\delta P\wedge\vartheta P\equiv 0.$$

But since the volume element $dP \times \delta P \wedge \vartheta P$ is arbitrary,

$$\operatorname{div}\mathbf{u}=\operatorname{div}'\mathbf{u}.$$

Existence.—The existence of div $\mathbf{u}$ depends on the existence of $d\mathbf{u}$, which in turn exists if

$$\operatorname*{Limit}_{h\to 0}\frac{\mathbf{u}(P+h\,dP)-\mathbf{u}(P)}{h}$$

exists.

Or, if $\mathbf{u}$ is a function of the distance r, we may write

$$\begin{aligned}\operatorname{div}\mathbf{u}\,d\tau &= \frac{\partial\mathbf{u}}{\partial r}dr\times\delta P\wedge\vartheta P+\frac{\partial\mathbf{u}}{\partial r}\delta r\times\vartheta P\wedge dP+\cdots\\ &= \frac{\partial\mathbf{u}}{\partial r}(\operatorname{grad}r\times dP)\times\delta P\wedge\vartheta P+\cdots+\cdots\\ &= \operatorname{grad}r\times\left[\frac{\partial\mathbf{u}}{\partial r}\times\delta P\wedge\vartheta P\cdot dP+\frac{\partial\mathbf{u}}{\partial r}\times\vartheta P\wedge dP\cdot\delta P\right.\\ &\qquad\left.+\frac{\partial\mathbf{u}}{\partial r}\times dP\wedge\delta P\cdot\vartheta P\right]\\ &= \operatorname{grad}r\times\frac{\partial\mathbf{u}}{\partial r}\cdot dP\times\delta P\wedge\vartheta P,\end{aligned}$$

* "Transformations Linéaires, Burali-Forti et Marcolongo," p. 23, Article 3.

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since the vector $dP \times \delta P \wedge \vartheta P \cdot (\partial u/\partial r)$ is expressible linearly in terms of the vectors dP, δP, and ϑP, in the form given in square brackets. Hence,

$$\operatorname{div} \mathbf{u} = \operatorname{grad} r \times \frac{\partial \mathbf{u}}{\partial r}.$$

Thus div **u** is seen to exist if $\partial \mathbf{u}/\partial r$ exists. Or, in rectangular coördinates, we may say that the divergence of the vector **u** exists if the partial derivatives of **u**, with respect to x, y, and z, exist.

Example.—Let us apply the definition (1) to the vector point function $\mathbf{r} = P - O$, O being a fixed point. We thus have, since $d\mathbf{r} = dP$,

$$\operatorname{div} \mathbf{r} \cdot d\tau = dP \times \delta P \wedge \vartheta P + \delta P \times \vartheta P \wedge dP + \vartheta P \times dP \wedge \delta P.$$

Since each of the scalar triple products in the right member is the same and equal to $d\tau$, this right member becomes $3d\tau$, so that $\operatorname{div} \mathbf{r} = 3$.

PART II.

Green's Theorem.

A General Form.—Turning from the elements to the more advanced part of vector analysis, we find in the "Transformations Linéaires" of Burali-Forti and Marcolongo the divergence of a vector defined as the first invariant of the homography* $d\mathbf{u}/dP$, written

$$\operatorname{div} \mathbf{u} \equiv I_1 \left(\frac{d\mathbf{u}}{dP}\right).$$

From this definition for div **u**, the following theorem may be obtained:

$$(4) \qquad \operatorname{div} \alpha u = I_1 \left(\alpha \frac{d\mathbf{u}}{dP}\right) - \operatorname{grad} K\alpha \times \mathbf{u},$$

where $K\alpha$ is the conjugate of the homography α. We are now able to demonstrate the following very general form of Green's Theorem:

$$(5) \qquad \int I_1 \left(K\alpha \cdot \frac{d \operatorname{grad} \beta}{dP} - K\beta \frac{d \operatorname{grad} \alpha}{dP}\right) d\tau = \int (K\alpha \operatorname{grad} \beta - K\beta \operatorname{grad} \alpha) \times \mathbf{n}\, d\sigma,$$

where α and β are homographies.

* A homography is defined as any linear operator which transforms a vector into a vector.

In rectangular coördinates, the homography

$$c \equiv \left\| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right|$$

operating on the vector

$$\mathbf{u} = u_1 i + u_2 j + u_3 k$$

changes the vector u into the vector

$$\alpha \mathbf{u} = (a_{11}u_1 + a_{12}u_2 + a_{13}u_3)\mathrm{i} + (a_{21}u_1 + a_{22}u_2 + a_{23}u_3)\mathrm{j} + (a_{31}u_1 + a_{32}u_2 + a_{33}u_3)\mathrm{k}.$$

Proof.—If we substitute (4) in Gauss' Theorem (1), we will have

$$\int I_1\left(\alpha\frac{d\mathbf{u}}{dP}\right)d\tau - \int \alpha\mathbf{u}\times\mathbf{n}d\sigma = -\int \operatorname{grad} K\alpha\times\mathbf{u}\,d\tau. \tag{6}$$

If we now take $\mathbf{u} = \operatorname{grad} K\beta$, (6) becomes

$$\int I_1\left(\alpha\frac{d\operatorname{grad} K\beta}{dP}\right)d\tau - \int \alpha \operatorname{grad} K\beta\times\mathbf{n}\,d\sigma = -\int \operatorname{grad} K\alpha\times \operatorname{grad} K\beta\,d\tau, \tag{7}$$

a form of Green's Theorem. Since the right member of (7) is symmetric in α and β, the left member is likewise, so that α and β may be interchanged without affecting the truth of equation (7). Thus:

$$\int I_1\left(\beta\frac{d\operatorname{grad} \alpha}{dP}\right)d\tau - \int \beta \operatorname{grad} K\alpha\times\mathbf{n}\,d\sigma = -\int \operatorname{grad} K\alpha\times \operatorname{grad} K\beta\,d\tau. \tag{8}$$

Subtracting (8) from (7) and transposing, we have

$$\int I_1\left(\alpha\frac{d\operatorname{grad} K\beta}{dP} - \beta\frac{d\operatorname{grad} K\alpha}{dP}\right)d\tau = \int(\alpha\operatorname{grad} K\beta - \beta\operatorname{grad} K\alpha)\times\mathbf{n}\,d\sigma. \tag{9}$$

Equation (5) is true for any two homographies. It is therefore true if we replace α and β by $K\alpha$ and $K\beta$. If we make this substitution in (9) and remember that $KK\alpha = \alpha$ we obtain equation (5).

Special Forms.—The operators I_1 and K are linear operators, changing a homography into a homography. Thus

$$I_1(m\alpha) = mI_1(\alpha),$$

if m is a scalar point function. Also $K(m) = m$. If, then, we replace the homographies α and β by the scalar point functions φ and ψ, equation (5) reduces to

$$\int\left[\varphi I_1\left(\frac{d\operatorname{grad}\psi}{dP}\right) - \psi I_1\left(\frac{d\operatorname{grad}\varphi}{dP}\right)\right]d\tau = \int\varphi\operatorname{grad}\psi - \psi\operatorname{grad}\varphi)\times\mathbf{n}\,d\sigma.$$

If we denote div grad by Δ this last equation may be written

$$\int(\varphi\Delta\psi - \psi\Delta\varphi)d\tau = \int(\varphi\operatorname{grad}\psi - \psi\operatorname{grad}\varphi)\times\mathbf{n}\,d\sigma,$$

the well-known form of Green's Theorem, which appears as a special instance of the more general equation (5). Further if α is taken as unity and grad β as the vector $\mathbf{u}$, (5) reduces to Gauss' Theorem as it should.

Another theorem, closely allied to these, but which may better be classed with the theorems in a previous paper* by the writer, will be given without proof. The proof may follow the lines suggested by that paper.

THEOREM.—*If* α *is a homography symmetrical in* P *and* M *such that* $d\alpha/dM = -(d\alpha/dP)$ *and if* $\mathbf{u}$ *is independent of* P, *the point of integration, then*:

$$\int \left(\frac{d\alpha}{dP}\mathbf{u}\right)\mathbf{x}\, d\tau = -\int \mathbf{u} \times \mathbf{n}\, \alpha\mathbf{x}\, d\sigma - \int \alpha \frac{d\mathbf{x}}{dP}\mathbf{u}\, d\tau.$$

* *Bulletin American Mathematical Society*, Vol. 22, Jan., 1916, p. 174. "Transformations in the Theory of the Linear Vector Function."

AN EXTENSION OF THE STURM-LIOUVILLE EXPANSION.

By Chester Claremont Camp.

Introduction.

In 1836–7 C. Sturm's formal development of a more or less arbitrary function $f(x)$ in terms of solutions of the self-adjoint equation

$$\frac{d}{dx}\left(k\frac{du}{dx}\right) + (\lambda g - l)u = 0$$

and the Sturmian boundary conditions

$$\begin{aligned} \alpha u(a) + \alpha' u'(a) &= 0, \\ \beta u(b) + \beta' u'(b) &= 0, \end{aligned} \qquad \left.\begin{aligned} |\alpha| + |\alpha'| &\neq 0 \\ |\beta| + |\beta'| &\neq 0 \end{aligned}\right\}$$

was considered by J. Liouville,* who undertook the problem of showing that the series converges and that its value is $f(x)$. His work is important although it did not satisfy all the requirements of modern mathematical rigor, but in two remarkable papers A. Kneser† completely settled all the more fundamental questions concerning the development. It remained for Haar‡ several years later to give the solution finality.

In 1908 Birkhoff || extended the theory not only to equations of the nth order of the form

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + \cdots + p_{n-1}\frac{du}{dx} + (p_n + \lambda g)u = 0,$$

but to systems no longer self-adjoint and conditions no longer Sturmian.

The theory is capable of extension in several directions. Bôcher§ considered a system of two homogeneous linear differential equations of the first order and studied the roots of a solution without regard to boundary conditions. Schlesinger¶ took a system of n linear equations of the first order with coefficients which are rational in x and obtained the asymptotic forms for a solution.

The object of this paper is to discuss an extension of a problem recently considered by Professor Hurwitz, namely the simultaneous expansion of two

* *Liouville's Journal*, Vol. 1 (1836), p. 253; Vol. 2 (1837), p. 16 and p. 418.

† *Math. Ann.*, Vol. 58, p. 81 and Vol. 60, p. 402.

‡ Goettingen dissertation (1909) reprinted in *Math. Ann.*, Vol. 69 (1910), p. 331. Also a second paper, *Math. Ann.*, Vol. 71 (1911), p. 38. See also Mercer, *Phil. Trans.*, Vol. 211 (1911), p. 111.

|| *Trans. Amer. Math. Soc.*, Vol. 9, p. 373.

§ *Trans. Amer. Math. Soc.*, Vol. 3 (1902), p. 196.

¶ *Math. Ann.*, Vol. 63 (1907), p. 277.

tinuous second derivatives for $0 \leqq x \leqq 1$; and the coefficients of u, v, in (2) are real constants such that the matrix*

$$\left\| \begin{matrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{matrix} \right\|$$

is of rank two and

$$(\alpha\beta) = (\gamma\delta) \neq 0 \tag{3}$$

where

$$(\alpha\beta) \equiv \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}.$$

The solution $u(x) \equiv v(x) \equiv 0$ is called the *trivial* solution of (1), (2). All other solutions are termed *non-trivial.*

Studying non-trivial solutions of (1), (2) we derive the following lemmas.

LEMMA I: *If* (u_m, v_m) *and* (u_n, v_n) *are two solutions of*

$$N_1(u_k v_k \lambda_k) = N_2(u_k v_k \lambda_k) = 0,$$

then

$$\left[u_m(x)v_n(x) - u_n(x)v_m(x)\right]_{x=0}^{x} = (\lambda_m - \lambda_n)\int_0^x\left[u_m(s)u_n(s) + v_m(s)v_n(s)\right]ds. \tag{4}$$

By hypothesis

$$\begin{aligned} u_m'(x) - [\lambda_m + a(x)]v_m(x) &= 0, \\ v_m'(x) + [\lambda_m + b(x)]u_m(x) &= 0, \\ u_n'(x) - [\lambda_n + a(x)]v_n(x) &= 0, \\ v_n'(x) + [\lambda_n + b(x)]u_n(x) &= 0. \end{aligned}$$

If we multiply these equations by $v_n(x)$, $-u_n(x)$, $-v_m(x)$, and $u_m(x)$ respectively, add, and integrate from 0 to x, we obtain the equation (4) required.

LEMMA II: *If* (u, v) *satisfies* $N_1 = \varphi$, $N_2 = \psi$, *where* φ, ψ *are functions of* x *continuous* $0 \leqq x \leqq 1$, *and* (U, V) *satisfies* $N_1 = N_2 = 0$ *for the same value of* λ, *then*

$$\left[u(x)V(x) - U(x)v(x)\right]_0^x = \int_0^x\left[V(s)\varphi(s) - U(s)\psi(s)\right]ds. \tag{5}$$

By hypothesis

$$\begin{aligned} u'(x) - [\lambda + a(x)]v(x) &= \varphi(x), \\ v'(x) + [\lambda + b(x)]u(x) &= \psi(x), \\ U'(x) - [\lambda + a(x)]V(x) &= 0, \\ V'(x) + [\lambda + b(x)]U(x) &= 0. \end{aligned}$$

* If the matrix is of rank less than two the conditions (2) are not independent. The condition $(\alpha\beta) = (\gamma\delta)$ is somewhat analogous to the condition of self-adjointness for single equations of order two. The case in which $(\alpha\beta) = (\gamma\delta) = 0$ can obviously be reduced to the problem considered by Professor W. A. Hurwitz.

Multiplying respectively by $V(x)$, $-U(x)$, $-v(x)$, $u(x)$; adding and integrating as before we get equation (5) above.

LEMMA III: *Two pairs of functions $u(x)$, $v(x)$; $U(x)$, $V(x)$; satisfying $U_1 = U_2 = 0$, satisfy also the relation*

$$[u(x)V(x) - v(x)U(x)]_0^1 = 0 \tag{6}$$

when (3) *is satisfied.*

We are given

$$\begin{aligned} \alpha_1 u(0) + \beta_1 v(0) + \gamma_1 u(1) + \delta_1 v(1) &= 0, \\ \alpha_2 u(0) + \beta_2 v(0) + \gamma_2 u(1) + \delta_2 v(1) &= 0, \\ \alpha_1 U(0) + \beta_1 V(0) + \gamma_1 U(1) + \delta_1 V(1) &= 0, \\ \alpha_2 U(0) + \beta_2 V(0) + \gamma_2 U(1) + \delta_2 V(1) &= 0. \end{aligned}$$

Since $(\alpha\beta) \neq 0$,

$$u(0) = -\frac{1}{(\alpha\beta)} \begin{vmatrix} \gamma_1 u(1) + \delta_1 v(1), & \beta_1 \\ \gamma_2 u(1) + \delta_2 v(1), & \beta_2 \end{vmatrix},$$

and

$$v(0) = -\frac{1}{(\alpha\beta)} \begin{vmatrix} \alpha_1, & \gamma_1 u(1) + \delta_1 v(1) \\ \alpha_2, & \gamma_2 u(1) + \delta_2 v(1) \end{vmatrix},$$

also $U(0)$, $V(0)$ are similar functions of $U(1)$, $V(1)$.

Substituting these values in (6) we get for the condition necessary that it be satisfied

$$(\alpha\beta)^2 = (\delta\beta)(\gamma\alpha) + (\gamma\beta)(\alpha\delta) \equiv (\alpha\beta)(\gamma\delta)$$

which is evidently true by (3).

LEMMA IV: *If (u_1, v_1), (u_2, v_2) are two linearly independent pairs of functions of x, continuous $0 \leqq x \leqq 1$, then*

$$G_2 \equiv \begin{vmatrix} (11) & (12) \\ (21) & (22) \end{vmatrix}$$

is real and greater than zero, where

$$(rs) \equiv \int_0^1 [u_r(s)\bar{u}_s(s) + v_r(s)\bar{v}_s(s)]ds$$

and $\bar{u}(x)$ represents the conjugate of $u(x)$.

Obviously $\bar{G}_2 = G_2$, i.e., G_2 is real.

Let

$$p(x) \equiv \begin{vmatrix} (11) & u_1(x) \\ (21) & u_2(x) \end{vmatrix},$$

$$q(x) \equiv \begin{vmatrix} (11) & v_1(x) \\ (21) & v_2(x) \end{vmatrix},$$

i.e.,

$$\begin{aligned} p(x) &\equiv c_1 u_1(x) + c_2 u_2(x), \\ q(x) &\equiv c_1 v_1(x) + c_2 v_2(x). \end{aligned}$$

Calculate

$$\int_0^1[p(x)\bar{u}_k(x) + q(x)\bar{v}_k(x)]dx \equiv \begin{vmatrix} (11) & (1k) \\ (21) & (2k) \end{vmatrix} = \begin{cases} 0, \text{ if } k = 1, \\ G_2, \text{ if } k = 2. \end{cases}$$

Then

$$\begin{aligned}\int_0^1(|p(x)|^2 + |q(x)|^2)dx &\equiv \int_0^1[p(x)\bar{p}(x) + q(x)\bar{q}(x)]dx \\ &\equiv \bar{c}_1\int_0^1[p(x)\bar{u}_1(x) + q(x)\bar{v}_1(x)]dx + \bar{c}_2\int_0^1[p(x)\bar{u}_2(x) + q(x)\bar{v}_2(x)]dx \\ &\equiv \bar{c}_2 G_2 \geqq 0.\end{aligned}$$

If $G_2 = 0$ and $\bar{c}_2 = c_2 \neq 0$, then $p(x) \equiv q(x) \equiv 0$ and (u_1, v_1), (u_2, v_2) will be linearly dependent.

If $G_2 = 0$ and $\bar{c}_2 = 0$, then since

$$c_2 \equiv (11) \equiv \int_0^1(|u_1(x)|^2 + |v_1(x)|^2)dx = 0,$$

$u_1 \equiv v_1 \equiv 0$ and they will again be linearly dependent, thus violating the hypothesis.

Hence

$$G_2 > 0 \tag{7}$$

since $\bar{c}_2 > 0$, and the lemma is proved.*

LEMMA V: *If* (3) *is satisfied, then*

$$[(\gamma\delta) + (\alpha\beta)]^2 \leqq [(\alpha\gamma) + (\beta\delta)]^2 + [(\alpha\delta) + (\gamma\beta)]^2. \tag{8}$$

(8) can easily be shown to be equivalent to

$$(\gamma\delta)^2 + (\alpha\beta)^2 \leqq (\alpha\gamma)^2 + (\beta\delta)^2 + (\alpha\delta)^2 + (\gamma\beta)^2. \tag{9}$$

But

$$(\alpha\gamma)^2 + (\beta\delta)^2 \geqq 2(\alpha\gamma)(\beta\delta),$$

and

$$(\alpha\delta)^2 + (\beta\gamma)^2 \geqq - 2(\alpha\delta)(\beta\gamma).$$

Hence the right member of (9) is

$$\begin{aligned}&\geqq 2(\alpha\gamma)(\beta\delta) - (\alpha\delta)(\beta\gamma) \\ &\geqq 2(\alpha\beta)(\gamma\delta), \text{ or by (3)} \\ &\geqq (\alpha\beta)^2 + (\gamma\delta)^2.\end{aligned}$$

Since (9) is true, (8) is also.

LEMMA VI: *If* (3) *holds, it is impossible for the coefficients in* (2) *to satisfy*

$$\begin{aligned}(\alpha\gamma) + (\beta\delta) &= 0, \\ (\alpha\delta) + (\gamma\beta) &= 0.\end{aligned} \tag{10}$$

* The lemma admits of the following generalization: If (u_1, v_1), (u_2, v_2), $\cdots (u_n, v_n)$ are n linearly independent pairs of functions of x, continuous $0 \leqq x \leqq 1$, then

$$G_n \equiv \begin{vmatrix} (11) & (12) & \cdots & (1n) \\ (21) & (22) & \cdots & (2n) \\ \cdot & \cdot & \cdot & \cdot \\ (n1) & (n2) & \cdots & (nn) \end{vmatrix}$$

is real and greater than zero, n being a positive integer.

By (3)

$$(\alpha\beta) - (\gamma\delta) = 0.$$

If then we assume (10) true, by squaring and adding all these equations, we have

$$(\alpha\gamma)^2 + (\beta\delta)^2 + (\alpha\delta)^2 + (\gamma\beta)^2 + (\alpha\beta)^2 + (\gamma\delta)^2 - 2(\alpha\gamma)(\beta\delta) \\ + 2(\alpha\delta)(\gamma\beta) - 2(\alpha\beta)(\gamma\delta) = 0.$$

The sum of the last three terms of the first member vanishes identically so that if (10) holds, each of the six determinants must vanish. Since this violates (3) the proof is complete.

Section II. Properties of Solutions of the Homogeneous and Non-homogeneous Systems.

LEMMA I: *A necessary and sufficient condition for the existence of a solution* (u, v) *of* (1), (2) *is that the determinant*

$$D(\lambda) \equiv \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_1v_1) & U_2(u_2v_2) \end{vmatrix} \tag{11}$$

vanish for some value of λ, *where* (u_1, v_1), (u_2, v_2) *are solutions of* (1) *defined by*

$$\begin{cases} u_1(0) = 1, & v_1(0) = 0, \\ u_2(0) = 0, & v_2(0) = 1. \end{cases} \tag{12}$$

By the existence theorem we know that a solution either of (1) or of the corresponding non-homogeneous system

$$N_1 = \varphi, \qquad N_2 = \psi,$$

where φ, ψ are functions of x, continuous $0 \leqq x \leqq 1$, will be entire in λ, provided it is defined by

$$\begin{cases} u(0) = c_1, \\ v(0) = c_2, \end{cases}$$

in which the c's are independent of λ. Clearly $D(\lambda)$ will also be entire in λ.

Any solution of (1) is expressible in the form

$$\begin{cases} u(x) = u(0)u_1(x) + v(0)u_2(x), \\ v(x) = u(0)\,v_1(x) + v(0)\,v_2(x). \end{cases} \tag{13}$$

In order that $u(x)$, $v(x)$ satisfy (2) it is necessary and sufficient that $u(0)$, $v(0)$ be determined so as to satisfy

$$\begin{cases} U_1(u) \equiv U_1(u_1v_1)u(0) + U_1(u_2v_2)v(0) = 0, \\ U_2(u) \equiv U_2(u_1v_1)u(0) + U_2(u_2v_2)v(0) = 0, \end{cases} \tag{14}$$

or if $u(x)$, $v(x)$ is to be a non-trivial solution, the determinant of coefficients of $u(0)$, $v(0)$, which is $D(\lambda)$, must vanish. This is obviously also sufficient.

LEMMA II: *The roots of $D(\lambda)$ are all real and the solutions of (1), (2) may be taken as real.*

Let $\lambda = \lambda_k$ be a root of $D(\lambda)$. Then (u_k, v_k) satisfies $N_1(u_k v_k \lambda_k) = 0$, $N_2(u_k v_k \lambda_k) = 0$ and $(\bar{u}_k, \bar{v}_k)$ will satisfy the same conditions for $\lambda = \bar{\lambda}_k$. Hence by Lemma I, Section I

$$[\bar{u}_k v_k - u_k \bar{v}_k]_0^x = (\bar{\lambda}_k - \lambda_k) \int_0^x (|u_k(s)|^2 + |v_k(s)|^2) ds.$$

Moreover $(\bar{u}_k, \bar{v}_k)$ satisfies $U_1(u) = U_2(u) = 0$ so that by Lemma III, Section I

$$(\bar{\lambda}_k - \lambda_k) \int_0^1 (|u_k|^2 + |v_k|^2) dx = 0.$$

If $\bar{\lambda}_k \neq \lambda_k$, (u_k, v_k) is a trivial solution which we have excluded. Therefore, $\bar{\lambda}_k = \lambda_k$ and λ_k is real.

Since each equation of the system (1), (2) is linear and $a(x)$, $b(x)$ are real, any solution of the system if complex may be broken up into real and imaginary parts which will separately satisfy the same system. When $D(\lambda)$ vanishes the ratio of $u(0)$ to $v(0)$ or of $v(0)$ to $u(0)$ is determinable* and by the existence theorem the solution $u(x)$, $v(x)$ is then uniquely determined except for a constant factor. Hence the coefficients of the real and imaginary parts constitute two solutions which are linearly dependent. It is therefore clear that we may restrict ourselves to real solutions of the system.

A similar discussion will show that we need consider only real solutions of the system

$$N_1 = \varphi, \qquad N_2 = \psi, \tag{15}$$

where φ, ψ are real functions of x and λ is restricted to real values, a restriction which we shall henceforth make.

A value of λ which makes $D(\lambda)$ vanish is called a *principal parameter value* and the corresponding solution of the homogeneous system (1), (2) is known as a principal solution.

Since $D(\lambda)$ is an entire function it cannot have more than a finite number of roots in any finite interval of the λ-axis, so that we may designate the roots of $D(\lambda)$ by λ_n, $n = 0, \pm 1, \pm 2, \cdots$.

THEOREM I: *A sufficient condition for the existence of a solution $(\tilde{u}, \tilde{v})$ of (15), (2) analytic for all real finite values of λ is that*

$$\int_0^1 \begin{vmatrix} u_n(s) & \varphi(s) \\ v_n(s) & \psi(s) \end{vmatrix} ds = 0 \tag{16}$$

for every (u_n, v_n) satisfying (1), (2) for $\lambda = \lambda_n$, $n = 0, \pm 1, \pm 2, \cdots$.

We shall consider in turn three kinds of values of λ:

Case I: Those for which the matrix of (11) is of rank two;

* If the matrix of (11) is of rank zero (Case III below), then this is found by the evaluation of an indeterminate form.

Case II: Those for which the matrix is of rank one (at $\lambda = \lambda_n$);

Case III: Those for which it is of rank zero (at $\lambda = \lambda_k$).

Take a solution of (1),

$$\begin{cases} u(x) = c_1 u_1(x) + c_2 u_2(x), \\ v(x) = c_1 v_1(x) + c_2 v_2(x), \end{cases} \tag{17}$$

which is identical with (13) provided $u(0) = c_1$, $v(0) = c_2$. This solution will satisfy (2) if (14) is satisfied and will be non-trivial provided c_1, c_2 are not both zero. For Case I at least two of the elements of $D(\lambda)$ must be different from zero. Without loss of generality assume that $U_1(u_2v_2) \neq 0$, then by continuity it will not vanish for values of λ nearby. Put $c_1 \equiv - U_1(u_2v_2)$, $c_2 \equiv U_1(u_1v_1)$. Then $U_1(uv) \equiv 0$, and $U_2(uv) \equiv D(\lambda)$.

Define another solution (U, V) of (1) by

$$\begin{cases} U(x) \equiv - c_2 u_1(x) + c_1 u_2(x), \\ V(x) \equiv - c_2 v_1(x) + c_1 v_2(x). \end{cases} \tag{18}$$

Then (u, v), (U, V) will be linearly independent since

$$W \equiv \begin{vmatrix} u(0) & v(0) \\ U(0) & V(0) \end{vmatrix} = \begin{vmatrix} c_1 & c_2 \\ - c_2 & c_1 \end{vmatrix} = c_1^2 + c_2^2 > 0.$$

Consider a solution $(\tilde{u}, \tilde{v})$ of (15) defined by

$$\begin{cases} \tilde{u}(x) \equiv u_0(x) + b_1 u(x) + b_2 U(x), \\ \tilde{v}(x) = v_0(x) + b_1 v(x) + b_2 V(x), \end{cases} \tag{19}$$

where (u_0, v_0) is a particular solution of (15) such that

$$u_0(0) = v_0(0) = 0.$$

Then

$$\begin{cases} U_1(\tilde{u}\tilde{v}) = U_1(u_0v_0) - b_2 W, \\ U_2(\tilde{u}\tilde{v}) = U_2(u_0v_0) + b_1 U_2(uv) + b_2 U_2(UV), \end{cases} \tag{20}$$

and $(\tilde{u}, \tilde{v})$ will satisfy (2) if

$$\begin{aligned} b_2 &= \frac{U_1(u_0v_0)}{W}, \\ b_1 &= \frac{- U_1(u_0v_0) U_2(UV) - U_2(u_0v_0) W}{W U_2(uv)}, \end{aligned} \tag{21}$$

$$\begin{aligned} \tilde{u}(x) &= u_0(x) + \frac{U_1(u_0v_0) U(x)}{W} - \frac{U_1(u_0v_0) U_2(UV) + U_2(u_0v_0) W}{W D(\lambda)} u(x), \\ \tilde{v}(x) &= v_0(x) + \frac{U_1(u_0v_0) V(x)}{W} - \frac{U_1(u_0v_0) U_2(UV) + U_2(u_0v_0) W}{W D(\lambda)} v(x), \end{aligned} \tag{22}$$

since $U_2(uv) = D(\lambda)$.

Hence if the matrix of (11) is of rank two, (22) determines a solution

$(\tilde{u}, \tilde{v})$ analytic in λ for that range for which $U_1(u_2v_2) \neq 0$. For other ranges of values of λ within which $U_1(u_2v_2)$ vanishes we choose in its place another element of the determinant $D(\lambda)$ which differs from zero throughout that range. This process determines b_1, b_2 in a different fashion. Clearly since the roots of $U_1(u_2v_2)$ and of every other element of $D(\lambda)$ are isolated, each element being entire in λ, the solution $(\tilde{u}, \tilde{v})$ determined in either way for points of the λ-axis at which $U_1(u_2v_2) \neq 0$ and some other element, e.g., $U_2(u_1v_1) \neq 0$, will be identical since there is one and only one such solution by the existence theorem. $\therefore$ The proof is complete for Case I.

Case II: Let us first show that no root of $D(\lambda)$ is double for this case. Assuming that $U_1(u_2v_2) \neq 0$ for $\lambda = \lambda_n$ and defining c_1, c_2 as before, we have by differentiating as to λ, since $u(x)$, $v(x)$ of (17) are entire in λ and satisfy (1):

$$\begin{cases} N_1(u_\lambda v_\lambda \lambda) = v(x), \\ N_2(u_\lambda v_\lambda \lambda) = -\, u(x). \end{cases} \tag{23}$$

If $D(\lambda)$ has a double root $\lambda = \lambda_n$, then for this value of λ

$$U_2(uv) = 0, \quad U_2(u_\lambda v_\lambda) = 0; \qquad N_1(uv\lambda) = N_2(uv\lambda) = 0;$$
$$U_1(uv) = 0, \quad U_1(u_\lambda v_\lambda) = 0.$$

Whence by Lemmas II, III, Section I

$$\int_0^1([u(s)]^2 + [v(s)]^2)ds = 0.$$

This is impossible since $u(0) \equiv - U_1(u_2v_2) \not\equiv 0$. Therefore $D(\lambda)$ has no double root.

For values sufficiently near λ_n, $D(\lambda) \neq 0$ and the solution $(\tilde{u}, \tilde{v})$ of (15), (2) analytic in λ is given by (22). And by Lemma II, Section I

$$[u_0(x)v(x) - u(x)v_0(x)]_0^x = \int_0^x[v(s)\varphi(s) - u(s)\psi(s)]ds, \tag{24}$$
$$[u_0(x)V(x) - U(x)v_0(x)]_0^x = \int_0^x[V(s)\varphi(s) - U(s)\psi(s)]ds. \tag{25}$$

The left members vanish at the lower limit since $u_0(0)$ and $v_0(0)$ are equal to zero, hence solving we obtain

$$u_0(x) = -\frac{1}{W}\int_0^x \begin{vmatrix} u(x) & U(x) & 0 \\ u(s) & U(s) & \varphi(s) \\ v(s) & V(s) & \psi(s) \end{vmatrix} ds, \tag{26}$$

$$v_0(x) = -\frac{1}{W}\int_0^x \begin{vmatrix} v(x) & V(x) & 0 \\ u(s) & U(s) & \varphi(s) \\ v(s) & V(s) & \psi(s) \end{vmatrix} ds. \tag{27}$$

If $\lambda \to \lambda_n$, $\tilde{u}(x)$ approaches the value

$$u_0(x) + \frac{U_1(u_0v_0)}{W}U(x) - \frac{u(x)}{W}\left\{\frac{\dfrac{\partial}{\partial\lambda}[U_1(u_0v_0)U_2(UV) + U_2(u_0v_0)W]}{D'(\lambda)}\right\}_{\lambda=\lambda_n} \tag{28}$$

provided only

$$U_1(u_0v_0)U_2(UV) - U_2(u_0v_0)W\rfloor_{\lambda=\lambda_n} = 0 \tag{29}$$

since $D'(\lambda) \neq 0$.

Again if (29) holds, $\bar{v}(x)$ will approach a limit as $\lambda \to \lambda_n$. In such a case $(\bar{u}, \bar{v})$ will be continuous if put equal to the limit approached at $\lambda = \lambda_n$ and therefore analytic for all λ in the interval.

Since $W = - U_1(UV)$, from (26), (27) it follows that (29) is equivalent to

$$\int_0^1 \left| \begin{matrix} \left\| \begin{matrix} \gamma_1 u(1)+\delta_1 v(1), & U_1(UV) \\ \gamma_2 u(1)+\delta_2 v(1), & U_2(UV) \end{matrix} \right\|, & \left| \begin{matrix} \gamma_1 U(1)+\delta_1 V(1), & U_1(UV) \\ \gamma_2 U(1)+\delta_2 V(1), & U_2(UV) \end{matrix} \right|, & 0 \\ u(s) & U(s) & \varphi(s) \\ v(s) & V(s) & \psi(s) \end{matrix} \right|_{\lambda=\lambda n} ds = 0. \tag{30}$$

The first element of the large determinant is the same as

$$\left| \begin{matrix} \gamma_1 u(1) + \delta_1 v(1), & \gamma_1 U(1) - \delta_1 V(1) \\ \gamma_2 u(1) + \delta_2 v(1), & \gamma_2 U(1) - \delta_2 V(1) \end{matrix} \right| - \left| \begin{matrix} \alpha_1 u(0) + \beta_1 v(0), & \alpha_1 U(0) + \beta_1 V(0) \\ \alpha_2 u(0) + \beta_2 v(0), & \alpha_2 U(0) + \beta_2 V(0) \end{matrix} \right|$$

since for $\lambda = \lambda_n$, $U_1(uv) = U_2(uv) = 0$, or simplifying further, this element reduces to $(\gamma\delta)W - (\alpha\beta)W = 0$.

Thus a sufficient condition for the existence of a solution $(\bar{u}, \bar{v})$ of (15), (2) analytic for all λ when $D(\lambda)$ is of rank one is that (16) be satisfied for all n such that $D(\lambda_n) = 0$, and the theorem is proved for Case II.

Case III. When every element of $D(\lambda)$ vanishes for some $\lambda = \lambda_k$, it is obvious that $D'(\lambda_k)$ also vanishes. Let us show that $D''(\lambda_k) \neq 0$ for such a case.

By the same reasoning as was used to prove $D'(\lambda) \neq 0$ for $\lambda = \lambda_n$ in Case II we show that if $U_1(u_{1\lambda}v_{1\lambda})$ and $U_2(u_{1\lambda}v_{1\lambda})$ both vanish at $\lambda = \lambda_k$, then

$$\int_0^1([u_1(x)]^2 + [v_1(x)]^2)dx = 0,$$

which is impossible since $u_1(0) = 1$. Hence $U_1(u_{1\lambda}v_{1\lambda})$ and $U_2(u_{1\lambda}v_{1\lambda})$ cannot both vanish at $\lambda = \lambda_k$. Similarly $U_1(u_{2\lambda}v_{2\lambda})$ and $U_2(u_{2\lambda}v_{2\lambda})$ cannot both vanish there. Without loss of generality we may assume

$$U_1(u_{2\lambda}v_{2\lambda}) \neq 0, \qquad \lambda = \lambda_k. \tag{31}$$

Now define a solution of (1) by

$$\begin{cases} u(x) \equiv u_1(x) + \dfrac{c_1}{c_2} u_2(x), \\ v(x) \equiv v_1(x) + \dfrac{c_1}{c_2} v_2(x), \end{cases} \tag{32}$$

and choose

$$\frac{c_1}{c_2} \equiv - \frac{U_1(u_1v_1)}{U_1(u_2v_2)}.$$

For values of λ sufficiently near λ_k, $U_1(u_2v_2) \neq 0$, otherwise (31) would be violated by Rolle's Theorem. Hence for all λ in the interval considered

$$U_1(uv) \equiv U_1(u_1v_1) - \frac{U_1(u_1v_1)}{U_1(u_2v_2)} U_1(u_2v_2) \equiv 0.$$

Consequently for all such λ, $U_1(u_\lambda v_\lambda) \equiv 0$.

Again

$$U_2(uv) = U_2(u_1v_1) - \frac{U_1(u_1v_1)}{U_1(u_2v_2)} U_2(u_2v_2) = \frac{-D(\lambda)}{U_1(u_2v_2)}, \qquad \lambda \neq \lambda_k.$$

At $\lambda = \lambda_k$,

$$U_2(uv) = -\frac{D'(\lambda)}{U_1(u_{2\lambda}v_{2\lambda})} = 0.$$

And

$$U_2(u_\lambda v_\lambda) = -\frac{D'(\lambda)}{U_1(u_2v_2)} + \frac{D(\lambda)U_1(u_{2\lambda}v_{2\lambda})}{[U_1(u_2v_2)]^2}, \qquad \lambda \neq \lambda_k.$$

For $\lambda = \lambda_k$,

$$U_2(u_\lambda v_\lambda) = -\frac{D''(\lambda)}{U_1(u_{2\lambda}v_{2\lambda})} + \frac{D(\lambda)U_1(u_{2\lambda\lambda}v_{2\lambda\lambda}) + D'(\lambda)U_1(u_{2\lambda})}{2U_1(u_2v_2)U_1(u_{2\lambda}v_{2\lambda})}.$$

This vanishes if $D''(\lambda) = 0$ at $\lambda = \lambda_k$. Applying the argument a third time we obtain

$$\int_0^1 (u^2 + v^2)dx = 0,$$

which is absurd since by (32) $u(0) = 1$.

$$\therefore \quad D''(\lambda) \neq 0 \text{ at } \lambda = \lambda_k.$$

Since for Case III when $\lambda = \lambda_k$ every solution of (1) satisfies (2), if we define a solution of (15) by

$$\begin{cases} \bar{u}(x) \equiv u_0(x) + b_1u_1(x) + b_2u_2(x), \\ \bar{v}(x) \equiv v_0(x) + b_1v_1(x) + b_2v_2(x), \end{cases} \tag{33}$$

where $u_0(0) = v_0(0) = 0$ defines a particular solution of (15), then $U_1(\bar{u}\bar{v}) = U_1(u_0v_0)$ and $U_2(\bar{u}\bar{v}) = U_2(u_0v_0)$. If these vanish b_1, b_2 will be arbitrary. Again by using results similar to (24), (25), (26), (27) we have, since

$$\begin{vmatrix} u_1(0) & v_1(0) \\ u_2(0) & v_2(0) \end{vmatrix} = 1,$$

$$U_1(u_0v_0) \equiv -\int_0^1 \begin{vmatrix} \gamma_1u_1(1) + \delta_1v_1(1), & \gamma_1u_2(1) + \delta_1v_2(1), & 0 \\ u_1(s), & u_2(s), & \varphi(s) \\ v_1(s), & v_2(s), & \psi(s) \end{vmatrix} ds,$$

$$U_2(u_0v_0) \equiv -\int_0^1 \begin{vmatrix} \gamma_2u_1(1) + \delta_2v_1(1), & \gamma_2u_2(1) + \delta_2v_2(1), & 0 \\ u_1(s), & u_2(s), & \varphi(s) \\ v_1(s), & v_2(s), & \psi(s) \end{vmatrix} ds.$$

For $\lambda = \lambda_k$, (u_1, v_1), (u_2, v_2) are solutions of (1), (2). Hence if we assume

that (16) holds for all solutions (u_n, v_n) of (1), (2), then $(\bar{u}, \bar{v})$ will satisfy (2) for b_1, b_2 arbitrary.

We wish to choose values for them such as to make the solution $(\bar{u}, \bar{v})$, now analytic $\lambda \neq \lambda_k$, continuous at $\lambda = \lambda_k$.

Since every solution of (1) may be expressed as a linear combination of u_1, u_2, v_1, v_2, we may put (33) in the form

$$\begin{cases} \bar{u}(x) \equiv u_0(x) + d_1 u(x) + d_2 u_2(x), \\ \bar{v}(x) \equiv v_0(x) + d_1 v(x) + d_2 v_2(x), \end{cases} \tag{34}$$

in which (u, v) is defined by (32) and c_1/c_2 has the same value as before. (u, v) and (u_2, v_2) are linearly independent since

$$\begin{vmatrix} u(0) & v(0) \\ u_2(0) & v_2(0) \end{vmatrix} = \begin{vmatrix} 1 & c_1/c_2 \\ 0 & 1 \end{vmatrix} \neq 0.$$

As $U_1(uv) \equiv 0$, we have

$$U_1(\bar{u}\bar{v}) = U_1(u_0v_0) + d_2 U_1(u_2v_2),$$

which will vanish for all λ in the interval considered if

$$d_2 \equiv -\frac{U_1(u_0v_0)}{U_1(u_2v_2)}.$$

Again

$$U_2(\bar{u}\bar{v}) = U_2(u_0v_0) + d_1 U_2(uv) + d_2 U_2(u_2v_2) = 0,$$

provided

$$d_1 = -\frac{U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)}{U_1(u_2v_2)U_2(uv)}.$$

For $\lambda \neq \lambda_k$,

$$U_2(uv) = -\frac{D(\lambda)}{U_1(u_2v_2)}$$

and

$$d_1 = \frac{U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)}{D(\lambda)}.$$

Hence for $\lambda = \lambda_k$ in a sufficiently small interval, $D(\lambda) \neq 0$ and $U_1(u_2, v_2) \neq 0$,

$$\bar{u}(x) = u_0(x) + \frac{U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)}{D(\lambda)} u(x) - \frac{U_1(u_0v_0)}{U_1(u_2v_2)} u_2(x). \tag{35}$$

If we let λ approach λ_k we get as the value approached by the right member:

$$u_0(x) - \frac{U_1(u_{0\lambda}v_{0\lambda})}{U_1(u_{2\lambda}v_{2\lambda})} u_2(x) + u(x) \frac{\frac{\partial^2}{\partial\lambda^2}[U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)]}{D''(\lambda)}, \quad \lambda = \lambda_k.$$

Evidently $u(x)$ approaches a limit since

$$\frac{c_1}{c_2} \to -\frac{U_1(u_{1\lambda}v_{1\lambda})}{U_1(u_{2\lambda}v_{2\lambda})}.$$

Also

$$\frac{\partial}{\partial\lambda}[U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)] = 0, \qquad \lambda = \lambda_k,$$

since each factor of both terms vanishes there. Also $D'(\lambda_k) = 0$.

Hence $\bar{u}(x)$ approaches a limit as $\lambda \to \lambda_k$, and similarly $\bar{v}(x)$ does also. Thus we have a solution $(\bar{u}, \bar{v})$ analytic for all λ in a small enough interval about $\lambda = \lambda_k$, which satisfies (2). By extending the reasoning as in Case I the proof is completed.

THEOREM II: *If for* $n = 0, \pm 1, \pm 2, \cdots$

$$\int_0^1 \begin{vmatrix} u_n(s) & \varphi(s) \\ v_n(s) & \psi(s) \end{vmatrix} ds = 0,$$

then $\varphi(x) \equiv \psi(x) \equiv 0$.

The proof will be merely outlined here since it is given in full in a recent article by Professor W. A. Hurwitz.

Let the solution of (15), (2) shown by the previous theorem to exist be represented by

$$\begin{aligned} u(x) &\equiv y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \cdots, \\ v(x) &\equiv z_0(x) + \lambda z_1(x) + \lambda^2 z_2(x) + \cdots. \end{aligned} \tag{36}$$

Since λ is restricted to real values we have but real functions of x with which to deal. Putting (36) in (15), (2) and equating coefficients of λ^k we get sets of differential equations and boundary conditions satisfied by (y_m, z_m), $m = 0, 1, 2, \cdots$. By eliminating $a(x)$, $b(x)$ from two different sets and using Lemma III, Section I, one gets

$$\textstyle\int_0^1(y_my_n + z_mz_n)dx = (\int_0^1 y_{m-1}y_{n+1} + z_{m-1}z_{n+1})dx = W_{m+n},$$

where

$$W_r \equiv \textstyle\int_0^1(y_ry_0 + z_rz_0)dx.$$

Then from the inequality

$$\begin{vmatrix} y_{m+1}(x) & y_{m+1}(\xi) \\ y_{m-1}(x) & y_{m-1}(\xi) \end{vmatrix}^2 + \begin{vmatrix} y_{m+1}(x) & z_{m+1}(\xi) \\ y_{m-1}(x) & z_{m-1}(\xi) \end{vmatrix}^2 + \begin{vmatrix} z_{m+1}(x) & y_{m+1}(\xi) \\ z_{m-1}(x) & y_{m-1}(\xi) \end{vmatrix}^2 + \begin{vmatrix} z_{m+1}(x) & z_{m+1}(\xi) \\ z_{m-1}(x) & z_{m-1}(\xi) \end{vmatrix}^2 \geqq 0, \tag{37}$$

for x and ξ in the interval from zero to one, by integrating as to x and ξ successively from 0 to 1 and simplifying, we have

$$W_{2m-2}W_{2m+2} - W_{2m}^2 \geqq 0. \tag{38}$$

Next comes the lemma that some $W_{2k} = 0$. Assume no $W_{2k} = 0$. Multiplying the series of (36) by $y_0(x)$, $z_0(x)$ respectively, adding and integrating

we obtain the uniformly convergent series

$$W_0 + \lambda W_1 + \lambda^2 W_2 + \cdots. \tag{39}$$

This converges absolutely as also does

$$|W_0| + \lambda^2 |W_2| + \lambda^4 |W_4| - \cdots. \tag{40}$$

Using (38), i.e.

$$\frac{W_0}{W_2} \geqq \frac{W_2}{W_4} \geqq \frac{W_4}{W_6} \geqq \cdots,$$

for $\lambda = \sqrt{\left|\frac{W_0}{W_2}\right|}$ in this we get an absurdity:

$|W_0| + |W_0| + |W_0| + \cdots$ convergent. Thus the lemma is proved, i.e.

$$\int_0^1 (y_k^2 + z_k^2) dx = 0, \qquad \text{or} \qquad y_k(x) \equiv z_k(x) \equiv 0. \tag{41}$$

Thence from the differential equations

$$y_{k-1}(x) \equiv z_{k-2}(x) \equiv \cdots \equiv y_0(x) \equiv 0,$$

and similarly for z and also $\psi(x) \equiv \varphi(x) \equiv 0$. Q.e.d.

Section III. Asymptotic Formulæ.

THEOREM III: For $|\lambda|$ large a solution of (1) defined by $u(0) = \alpha$, $v(0) = \beta$ takes the form

$$\begin{cases} u(x) = \alpha \cos \xi + \beta \sin \xi + O\left(\frac{1}{\lambda}\right), \\ v(x) = \beta \cos \xi - \alpha \sin \xi + O\left(\frac{1}{\lambda}\right), \end{cases} \tag{42}$$

and its partial derivatives, the form

$$\begin{cases} u_\lambda(x) = \quad \beta x \cos \xi - \alpha x \sin \xi + O\left(\frac{1}{\lambda}\right), \\ v_\lambda(x) = -\alpha x \cos \xi - \beta x \sin \xi + O\left(\frac{1}{\lambda}\right), \end{cases} \tag{43}$$

where

$$\xi \equiv \lambda x + \tfrac{1}{2}\int_0^x [a(s) + b(s)] ds. \tag{44}$$

I give an outline of a proof analogous to that used in Professor Hurwitz's recent article.

Assume

$$\begin{cases} u(x) \equiv U + \left(1 + \frac{a(x)}{\lambda}\right)(\alpha \cos \xi + \beta \sin \xi), \\ v(x) \equiv V + \left(1 + \frac{b(x)}{\lambda}\right)(\beta \cos \xi - \alpha \sin \xi). \end{cases} \tag{45}$$

Putting in (1) we get

$$\begin{cases} N_1(UV\lambda) = \dfrac{\varphi}{\lambda}, \\ N_2(UV\lambda) = \dfrac{\psi}{\lambda}, \end{cases} \tag{46}$$

where φ, ψ, φ_λ, ψ_λ are $O(1)$ as regards λ.

Using as multipliers $\cos \lambda x$, $-\sin \lambda x$, adding, integrating from 0 to x, and combining this with the result of repeating the procedure with $\sin \lambda x$, $\cos \lambda x$ as multipliers we get

$$\begin{cases} U(x) = \dfrac{F}{\lambda} + \int_0^x [K_{11}U(s) + K_{12}V(s)]ds, \\ V(x) = \dfrac{G}{\lambda} + \int_0^x [K_{21}U(s) + K_{22}V(s)]ds, \end{cases} \tag{47}$$

where F, G and the K's are $O(1)$, and F_λ, G_λ, K_λ are also.

$$\begin{aligned} \therefore\quad U_\lambda(x) &= \frac{H}{\lambda} + \int_0^x [K_{31}U(s) + K_{32}V(s) + K_{33}U_\lambda(s) + K_{34}V_\lambda(s)]ds, \\ V_\lambda(x) &= \frac{J}{\lambda} + \int_0^x [K_{41}U(s) + K_{42}V(s) + K_{43}U_\lambda(s) + K_{44}V_\lambda(s)]ds, \end{aligned} \tag{48}$$

in which H, J, and the K's of (48) are $O(1)$. By the process of successive approximations we obtain from (47), (48),

$$U(x) = O\left(\frac{1}{\lambda}\right), \quad V(x) = O\left(\frac{1}{\lambda}\right), \quad U_\lambda(x) = O\left(\frac{1}{\lambda}\right), \quad V_\lambda(x) = O\left(\frac{1}{\lambda}\right). \tag{49}$$

From (45), (49) we see that (42) is true. Also by differentiating (45) as to λ and using (49) the rest of the proof is obvious.

COROLLARY: For $|\lambda|$ large $D(\lambda)$, $D'(\lambda)$ take the forms

$$\begin{cases} D(\lambda) = \sqrt{A^2 + B^2} \sin [\xi(1) + \varphi] + C + O\left(\dfrac{1}{\lambda}\right), \\ D'(\lambda) = \sqrt{A^2 + B^2} \cos [\xi(1) + \varphi] + O\left(\dfrac{1}{\lambda}\right), \end{cases} \tag{50}$$

where

$$A \equiv (\alpha\gamma) + (\beta\delta), \qquad B \equiv (\alpha\delta) + (\gamma\beta), \qquad C = 2(\alpha\beta), \tag{51}$$

and φ is defined by

$$\cos \varphi \equiv \frac{A}{\sqrt{A^2 + B^2}}, \qquad \sin \varphi \equiv \frac{B}{\sqrt{A^2 + B^2}}. \tag{52}$$

By Lemma VI, Section I, A and B cannot both vanish. From Theorem III

$$\begin{cases} u_1(x) = \cos\xi + O\left(\dfrac{1}{\lambda}\right), \\ v_1(x) = -\sin\xi + O\left(\dfrac{1}{\lambda}\right), \\ u_2(x) = \sin\xi + O\left(\dfrac{1}{\lambda}\right), \\ v_2(x) = \cos\xi + O\left(\dfrac{1}{\lambda}\right). \end{cases} \tag{53}$$

Putting these in (11) we get

$$D(\lambda) = A\sin\xi(1) + B\cos\xi(1) + (\alpha\beta) + (\gamma\delta) + O\left(\frac{1}{\lambda}\right). \tag{54}$$

Again using the asymptotic values for $u_{1\lambda}(x)$, $v_{1\lambda}(x)$, etc., from (53) in the identity

$$D'(\lambda) \equiv \begin{vmatrix} U_1(u_{1\lambda}v_{1\lambda}) & U_1(u_{2\lambda}v_{2\lambda}) \\ U_2(u_1v_1) & U_2(u_2v_2) \end{vmatrix} + \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_{1\lambda}v_{1\lambda}) & U_2(u_{2\lambda}v_{2\lambda}) \end{vmatrix}$$

we derive

$$D'(\lambda) = A\cos\xi(1) - B\sin\xi(1) + O\left(\frac{1}{\lambda}\right),$$

and (50) follows by (3), (51), (52).

To show $D(\lambda)$ has roots, no matter how large $|\lambda|$ is, consider the

THEOREM IV: *If l_n is a principal parameter value for the system* (1) *and the boundary conditions*

$$\begin{cases} \alpha_1 u(0) + \beta_1 v(0) = 0, \\ \gamma_1 u(1) + \delta_1 v(1) = 0, \end{cases} \tag{55}$$

then there exist exactly two roots of $D(\lambda)$ in the intervals:

Case I. (l_{2p}, l_{2p+2}), $p = 0, \pm 1, \pm 2, \cdots$, if $P_{21} > 0$ for $\lambda = l_1$;

Case II. (l_{2p-1}, l_{2p+1}), $p = 0, \pm 1, \pm 2, \cdots$, if $P_{21} < 0$ for $\lambda = l_1$.*

Define

$$\begin{cases} B_1(x) \equiv \alpha_1 u(x) + \beta_1 v(x), \\ B_2(x) \equiv \alpha_2 u(x) + \beta_2 v(x), \\ P_1(x) \equiv -\gamma_1 u(x) - \delta_1 v(x), \\ P_2(x) \equiv -\gamma_2 u(x) - \delta_2 v(x), \end{cases} \tag{56}$$

and determine two solutions, $[u_1(x\lambda), v_1(x\lambda)]$, $[u_2(x\lambda), v_2(x\lambda)]$, of (1) such that

$$\begin{cases} B_{11}(0) = 0, & B_{21}(0) = 1, \\ B_{12}(0) = 1, & B_{22}(0) = 0; \end{cases} \tag{57}$$

in which the second subscript refers to the solution involved. Without

* I have followed the method of proof used by Professor Birkhoff in an article published in the *Transactions* in 1909 entitled, "Existence and Oscillation Theorem for a Certain Boundary Value Problem."

loss of generality we may assume that the coefficients α_1, β_1, γ_1, δ_1 have been divided by a constant such that

$$(\alpha\beta) = (\gamma\delta) = 1. \tag{58}$$

Then we prove the

LEMMA I: *At a simple value λ_n of λ, $D(\lambda)$ changes sign in such a way that $D'(\lambda)$ has the sign of $-P_{11}$ or P_{22}, where*

$$\begin{cases} P_{11} \equiv -\gamma_1 u_1(1) - \delta_1 v_1(1), \\ P_{22} \equiv -\gamma_2 u_2(1) - \delta_2 v_2(1). \end{cases} \tag{59}$$

The solutions (u_1, v_1), (u_2, v_2) defined by (57) are linearly independent since

$$\begin{vmatrix} u_1(0) & v_1(0) \\ u_2(0) & v_2(0) \end{vmatrix} = \begin{vmatrix} -\beta_1 & \alpha_1 \\ \beta_2 & -\alpha_2 \end{vmatrix} = (\alpha\beta) = -1 \tag{60}$$

and hence by a relation similar to Abel's

$$|u(x), \quad v(x)| = \text{constant} \neq 0.$$

Thus

$$B_{11}(x)B_{22}(x) - B_{12}(x)B_{21}(x) = -1, \qquad 0 \leqq x \leqq 1. \tag{61}$$

Likewise

$$P_{11}(x)P_{22}(x) - P_{12}(x)P_{21}(x) = -1, \qquad 0 \leqq x \leqq 1. \tag{62}$$

Also any solution of (1) may be written

$$u(x) = c_1 u_1(x) + c_2 u_2(x),$$
$$v(x) = c_1 v_1(x) + c_2 v_2(x).$$

If this is to satisfy (2) we must have

$$D(\lambda) = \begin{vmatrix} -P_{11}, & 1 - P_{12} \\ 1 - P_{21}, & -P_{22} \end{vmatrix} = 0. \tag{63}$$

We have shown that the necessary condition for a double value of λ is that $D(\lambda)$ be of rank zero, i.e.,

$$\begin{cases} P_{11} = P_{22} = 0, \\ P_{21} = P_{12} = 1. \end{cases} \tag{64}$$

This is obviously also sufficient.

Since

$$N_1(u_1 v_1 \lambda) = N_2(u_1 v_1 \lambda) = 0 \tag{65}$$

and

$$N_1(u_{1\lambda} v_{1\lambda} \lambda) = v_1, \qquad N_2(u_{1\lambda} v_{1\lambda} \lambda) = -u, \tag{66}$$

by Lemma II, Section I,

$$u_{1\lambda} v_1 - u_1 v_{1\lambda} = \int_0^x (u_1^2 + v_1^2) ds. \tag{67}$$

Similarly

$$u_{2\lambda} v_2 - u_2 v_{2\lambda} = \int_0^x (u_2^2 + v_2^2) ds. \tag{68}$$

Again from

$$N_1(u_2v_2\lambda) = N_2(u_2v_2\lambda) = 0 \tag{69}$$

and (66) by the same Lemma

$$u_{1\lambda}v_2 - u_2v_{1\lambda} = \int_0^x (u_1u_2 + v_1v_2)ds. \tag{70}$$

Similarly

$$u_{2\lambda}v_1 - u_1v_{2\lambda} = \int_0^x (u_1u_2 + v_1v_2)ds. \tag{71}$$

From (67), (70)

$$u_{1\lambda}(x) = \int_0^x[(u_1^2 + v_1^2)u_2(x) - (u_1u_2 + v_1v_2)u_1(x)]ds. \tag{72}$$

Similarly

$$v_{1\lambda}(x) = \int_0^x[(u_1^2 + v_1^2)v_2(x) - (u_1u_2 + v_1v_2)v_1(x)]ds. \tag{73}$$

Also from (68), (71)

$$u_{2\lambda}(x) = \int_0^x[(u_1u_2 + v_1v_2)u_2(x) - (u_2^2 + v_2^2)u_1(x)]ds, \tag{74}$$

$$v_{2\lambda}(x) = \int_0^x[(u_1u_2 + v_1v_2)v_2(x) - (u_2^2 + v_2^2)v_1(x)]ds. \tag{75}$$

Now from (62), (63)

$$D(\lambda) \equiv P_{12} + P_{21} - 2. \tag{76}$$

Hence by (72), (73), (74), (75), (76)

$$D'(\lambda) = P_{12\lambda} + P_{21\lambda}$$

or

$$D'(\lambda) = \int_0^1[P_{22}u_1^2 + (P_{12} - P_{21})u_1u_2 - P_{11}u_2^2 + P_{22}v_1^2 + (P_{12} - P_{21})v_1v_2 - P_{11}v_2^2]ds. \tag{77}$$

The integrand consists of two quadratic forms whose discriminant is

$$(P_{12} - P_{21})^2 + 4P_{11}P_{22} \tag{78}$$

or by (62)

$$(P_{12} + P_{21})^2 - 4. \tag{79}$$

This vanishes when $D(\lambda)$ does. For a simple root of $D(\lambda)P_{11}$, P_{22} cannot both vanish since then by (78) and (63), (64) would be satisfied. Again neither form can vanish because (u_1, v_1) and (u_2, v_2) are linearly independent. Thus the Lemma is proved.

LEMMA II: *At a double value of* λ, *say* λ_k, $D(\lambda)$ *maintains a negative sign.*

From (64)

$$\Delta P_{11} = \bar{P}_{11}, \quad \Delta P_{22} = \bar{P}_{22}, \quad 1 + \Delta P_{12} = \bar{P}_{12}, \quad 1 + \Delta P_{21} = \bar{P}_{21}, \tag{80}$$

in which

$$\bar{P} \equiv P(\lambda + \Delta\lambda), \qquad \lambda = \lambda_k. \tag{81}$$

Then from (62)

$$\bar{P}_{11}\bar{P}_{22} - \bar{P}_{12}\bar{P}_{21} = -1$$

or

$$\Delta P_{11}\Delta P_{22} - (1 + \Delta P_{12})(1 + \Delta P_{21}) = -1, \tag{82}$$

and by (76)

$$\Delta D = \Delta P_{12} + \Delta P_{21} = \Delta P_{11}\Delta P_{22} - \Delta P_{12}\Delta P_{21}. \tag{83}$$

Clearly

$$\Delta P_{11} \equiv -\gamma_1 \Delta u_1(1) - \delta_1 \Delta v_1(1),$$

and so for the others. Also

$$N_1(\Delta u, \Delta v, \lambda) = \Delta\lambda \bar{v},$$
$$N_2(\Delta u, \Delta v, \lambda) = -\Delta\lambda \bar{u}.$$

Using these for (u_1,v_1), (u_2,v_2) we get results for $\frac{\Delta u_1}{\Delta\lambda}, \frac{\Delta u_2}{\Delta\lambda}, \frac{\Delta v_1}{\Delta\lambda}, \frac{\Delta v_2}{\Delta\lambda}$ which differ little from the right members of equations (72) to (75). Putting these values in (83) and omitting in each term infinitesimals of order greater than two, we obtain

$$\Delta D = [\textstyle\int_0^1 (u_1u_2 + v_1v_2)\Delta\lambda ds]^2 - \int_0^1 (u_1^2 + v_1^2)\Delta\lambda ds \int_0^1 (u_2^2 + v_2^2)\Delta\lambda ds.$$

By Lemma IV, Section I, $\Delta D < 0$, since the solutions are real, and $D(\lambda)$ preserves a negative sign at $\lambda = \lambda_k$.

LEMMA III: *$D(\lambda)$ has the same sign as P_{21} at the values $\lambda = l_n$, $n = 0, \pm 1, \pm 2, \cdots$, unless $P_{21} = 1$, when $D(\lambda) = 0$.*

Since $u_1(0) = -\beta$, $v_1(0) = \alpha$ (Cf. (60)), (u_1,v_1) is the only solution except for a constant factor which satisfies (55), (1) for $\lambda = l_0, l_{\pm 1}, l_{\pm 2}, \cdots$ but for no others. (See Professor Hurwitz's article.) These values separate the λ-axis into the intervals

$$\cdots (l_{-n}, l_{-n+1}), \quad \cdots (l_{-1}, l_0), \quad (l_0, l_1), \quad \cdots (l_n, l_{n+1}), \quad \cdots \tag{84}$$

By (62) at $\lambda =$ one of these values, say l_r,

$$P_{12}P_{21} = 1. \tag{85}$$

And by (76)

$$D(l_r) = \frac{1}{P_{21}}(1 - P_{21})^2, \qquad r = 0, \pm 1, \pm 2, \cdots. \tag{86}$$

And the rest of the proof is evident.

Professor Hurwitz has shown that P_{11} has no double roots, also that for the system (1) and

$$\begin{cases} B_{11}(0) = 0, \\ P_{21}(1) = 0, \end{cases} \tag{87}$$

P_{21} has no double roots. Hence P_{11}, P_{21} change sign when they vanish. Again since

$$P_{11}P_{21\lambda} - P_{11\lambda}P_{21} \equiv u_1 v_{1\lambda} - v_1 u_{1\lambda}]_{x=1},$$

by (67) this cannot vanish but is negative. From these facts it is easy to show that the roots of P_{11}, P_{21} separate each other as well as do those of

$P_{11\lambda}$, $P_{21\lambda}$. Clearly then P_{21} alternates in sign at the values $\cdots l_{-2}, l_{-1}, l_0, l_1, l_2, \cdots$ and we have two cases:

$$\text{Case I: } \left\{ \begin{array}{l} D(\lambda) \geqq 0 \text{ at } \lambda = l_{\pm 1}, l_{\pm 3}, \cdots \\ D(\lambda) < 0 \text{ at } \lambda = l_0, \quad l_{\pm 2}, \cdots \end{array} \right\} \text{ when } P_{21} > 0 \text{ for } \lambda = l_1;$$

$$\text{Case II: } \left\{ \begin{array}{l} D(\lambda) < 0 \text{ at } \lambda = l_{\pm 1}, l_{\pm 3}, \cdots \\ D(\lambda) \geqq 0 \text{ at } \lambda = l_0, \quad l_{\pm 2}, \cdots \end{array} \right\} \text{ when } P_{21} < 0 \text{ for } \lambda = l_1.$$

There must obviously exist roots of $D(\lambda)$ as follows:

In Case I—at least two values λ_n, λ_{n+1} in each double interval

$$(l_{2p}, l_{2p+2}), \qquad p = 0, \pm 1, \pm 2, \cdots,$$

such that

$$l_{2p} < \lambda_n \leqq l_{2p+1} \leqq \lambda_{n+1} < l_{2p+2} \tag{88}$$

and at least one such that

$$l_1 \leqq \lambda_r < l_2.$$

In Case II—at least two values λ_m, λ_{m+1} in each double interval

$$(l_{2p-1}, l_{2p+1}), \qquad p = 0, \pm 1, \pm 2, \cdots,$$

such that

$$l_{2p-1} < \lambda_m \leqq l_{2p} \leqq \lambda_{m+1} < l_{2p+1}. \tag{89}$$

To show that there are exactly two roots in (l_{2p}, l_{2p+2}) we make use of the separative property of the roots of P_{11}, P_{21} and their derivatives. Obviously there must be an even number of roots, since we count a double root as two. If a double root occurs it must fall at l_{2p+1} by (64) and Lemma III. Then by Lemmas II and I there can be no root elsewhere. If there is no double root, there must be less than four roots, for otherwise there would be at least two in one of the intervals (l_{2p}, l_{2p+1}), (l_{2p+1}, l_{2p+2}), which violates Lemma I.

Similarly we can show that there is exactly one root in the interval (l_1, l_2) and exactly two for Case II in the intervals (l_{2p-1}, l_{2p+1}) provided we count a double root at l_1 once each in the intervals (l_0, l_1), (l_1, l_2). Thus the Theorem is proved.

COROLLARY. *For* $|\lambda|$ *large,* $O(1/\lambda_n) = O(1/n\pi)$, $|n|$ *large.*

Professor Hurwitz has shown that for $|\lambda|$ large, $l_n = n\pi + \theta_1 + O(1/n)$, where θ_1 is a constant. Thus the intervals (l_n, l_{n+2}) for $|\lambda|$, $|n|$ large are of length 2π to within $O(1/n)$. In each of these intervals for n odd or even according to Case I or II by the Theorem there exist just two values λ_m, λ_{m+1} of λ which are roots of $D(\lambda)$. Since in any finite interval of the λ-axis there are but a finite number of roots l_r or λ_m, for $|n|$ large enough we have

$$\lambda = l_{n-\varepsilon} - k, \qquad 0 \leqq |k| \leqq \pi,$$

where s is a finite integer positive or negative, provided we begin to count the l's and λ's so that $l_0 \geqq 0$, $\lambda_0 \geqq 0$ and $l_{-1} < 0$, $\lambda_{-1} < 0$. Clearly then

$$\lambda_n = n\pi + s\pi - k + \theta_1 + O\left(\frac{1}{n}\right) \tag{90}$$

and since $n\pi$ is the dominant part,

$$O\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{n\pi}\right); \text{ q.e.d.}$$

THEOREM V: *If* (u_n, v_n) *corresponds to* λ_n *defined as in the previous Corollary, then this solution of* (1), (2) *takes the asymptotic form*

$$\begin{cases} u_n(x) = \sin\left[n\pi x + P(x)\right] + O\left(\frac{1}{n}\right), \\ v_n(x) = \cos\left[n\pi x + P(x)\right] + O\left(\frac{1}{n}\right), \end{cases} \tag{91}$$

when $|n|$ *is large.*

By the Corollary of Theorem III $D(\lambda)$ will have a root when

$$\sin\left[\xi(1) + \varphi\right] = -\frac{C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right) \tag{92}$$

or when

$$\xi(1) + \varphi = \sin^{-1}\frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{n}\right), \tag{93}$$

since by Lemma V, Section I, $|C| \leqq \sqrt{A^2 + B^2}$, and by Theorem IV the right member of (92) must be numerically $\leqq 1$, also

$$\sin^{-1}\left[\frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right)\right] = \sin^{-1}\frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right)$$

and for $\lambda = \lambda_n$,

$$O\left(\frac{1}{\lambda}\right) = O\left(\frac{1}{n\pi}\right) = O\left(\frac{1}{n}\right).$$

Hence from (93), since by (58) $C > 0$, $D(\lambda) = 0$ when

$$\lambda + \int_0^1 \frac{a(x) + b(x)}{2}\,dx + \varphi = \sin^{-1}\frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{n}\right).$$

and for m odd

$$\begin{cases} \lambda_n = m\pi + \psi - \int_0^1 \frac{a(x) + b(x)}{2}\,dx - \varphi + O\left(\frac{1}{n}\right), \\ \lambda_{n+1} = (m+1)\pi - \psi - \int_0^1 \frac{a(x) + b(x)}{2}\,dx - \varphi + O\left(\frac{1}{n}\right), \end{cases} \tag{94}$$

in which ψ is the angle in radian measure between 0 and $\pi/2$ inclusive

whose sine is $|C|/\sqrt{A^2+B^2}$. Clearly (94) may be expressed as

$$\lambda_n = m\pi + (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2}\,dx - \varphi + O\left(\frac{1}{n}\right), \text{ all } n, \tag{95}$$

where $|n|$, $|\lambda_n|$ are large and $m \equiv n + s$. From (42)

$$u(x) = \sin(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

$$v(x) = \cos(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

provided we choose

$$\alpha \equiv -\sin\theta_0, \qquad \beta \equiv \cos\theta_0.$$

This is always possible if we divide $u(0)$, $v(0)$ by $\sqrt{[u(0)]^2 + [v(0)]^2}$ and call the quotients α, β respectively. Then by (95)

$$u_n(x) = \sin\left[m\pi x + \theta_2 x + \tfrac{1}{2}\int_0^x [a(s) + b(s)]ds - \theta_0\right] + O\left(\frac{1}{n}\right),$$

$$v_n(x) = \cos\left[m\pi x + \theta_2 x + \tfrac{1}{2}\int_0^x [a(s) + b(s)]ds - \theta_0\right] + O\left(\frac{1}{n}\right),$$

where

$$\theta_2 \equiv (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2}\,dx - \varphi$$

and the formulæ (91) follow, provided we define

$$P(x) \equiv \theta_2 x + s\pi x + \int_0^x \frac{a(s)+b(s)}{2}\,ds - \theta_0,$$

since $m \equiv n + s$, $\theta_2 \equiv -k + \theta_1$ as shown by (90).

COROLLARY: *If we normalize* (u_n, v_n), *it will have the same form* (91).

Section IV. Expansion Theorems.

THEOREM VI: *If two otherwise arbitrary functions* $f(x)$, $g(x)$ *satisfy* (2) *and have continuous second derivatives,* $0 \leqq x \leqq 1$, *then they are expansible in the form*

$$f(x) \equiv \sum_{n=-\infty}^{+\infty} c_n u_n(x), \qquad g(x) = \sum_{n=-\infty}^{+\infty} c_n v_n(x), \tag{96}$$

where

$$c_n \equiv \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx, \tag{97}$$

and (u_n, v_n) *represents the set of normal orthogonal solutions of* (1), (2).

In case λ_k *is a double value we count it as two and take any two linearly independent normal orthogonal solutions for that value of* λ.

Let us first show that two solutions (u_m, v_m), (u_n, v_n) of (1), (2) for different values of λ are orthogonal, i.e.,

$$\int_0^1 [u_m(s)u_n(s) + v_m(s)v_n(s)]ds = 0. \tag{98}$$

It is only necessary to apply Lemmas I, III, Section I, and divide by $\lambda_m - \lambda_n$. Obviously they are easily normalized by dividing in the case of (u_m, v_m) by $\sqrt{\int_0^1 (u_m^2 + v_m^2)dx}$.

If we assume the series of (96) to be uniformly convergent, then

$$\int_0^1 [f(x)u_m(x) + g(x)v_m(x)]dx = c_m$$

and the series become

$$\left\{\begin{array}{l} \sum\limits_{n=-\infty}^{+\infty} u_n(x)\int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx, \\ \sum\limits_{n=-\infty}^{+\infty} v_n(x)\int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx. \end{array}\right. \tag{99}$$

To study these we assume $|\lambda_n| > |a(x)|$, also $> |b(x)|$. We have

$$\begin{aligned} u_n'(x) - (\lambda_n + a)v_n(x) &= 0, \\ v_n'(x) + (\lambda_n + b)u_n(x) &= 0. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 f(x)u_n(x)dx &= -\int_0^1 \frac{f(x)}{\lambda_n + b} v_n'(x)dx \\ &= \left[-\frac{f(x)v_n(x)}{\lambda_n + b}\right]_0^1 + \int_0^1 v_n(x)\frac{d}{dx}\left[\frac{f(x)}{\lambda_n + b(x)}\right]dx \\ &= \left[-\frac{fv_n}{\lambda_n + b}\right]_0^1 + \int_0^1 \frac{u_n'(x)}{\lambda_n + a}\frac{d}{dx}\left[\frac{f}{\lambda_n + b}\right]dx. \end{aligned}$$

But since by the Corollary of Theorem IV $O\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{n\pi}\right) = O\left(\frac{1}{n}\right)$, for $|n|$ large, we have

$$\frac{\dfrac{d}{dx}\left(\dfrac{f(x)}{\lambda_n + b(x)}\right)}{\lambda_n + a(x)} = \frac{\dfrac{f'(x)}{\lambda_n + b} - \dfrac{f(x)b'(x)}{[\lambda_n + b]^2}}{\lambda_n + a} = O\left(\frac{1}{n^2}\right).$$

$$\therefore \int_0^1 fu_n dx = \left[-\frac{fv_n}{\lambda_n + b}\right]_0^1 + O\left(\frac{1}{n^2}\right).$$

Similarly

$$\int_0^1 gv_n dx = \left[\frac{gu_n}{\lambda_n + a}\right]_0^1 + O\left(\frac{1}{n^2}\right).$$

Hence

$$c_n = \left[\frac{gu_n}{\lambda_n + a} - \frac{fv_n}{\lambda_n + b}\right]_0^1 + O\left(\frac{1}{n^2}\right)$$

or

$$c_n = \frac{1}{\lambda_n}[gu_n - fv_n]_0^1 + O\left(\frac{1}{n^2}\right). \tag{100}$$

Since by hypothesis f, g satisfy

$$U_1(f, g) = U_2(f, g) = 0, \tag{101}$$

by Lemma III, Section I, $c_n = O(1/n^2)$ and each series in (99) converges uniformly. If $f(x)$, $g(x)$ are expansible, we get the series (99) for them. To show that these series converge uniformly to $f(x)$, $g(x)$ respectively, we define

$$F(x) \equiv \sum_{n=-\infty}^{+\infty} u_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx,$$
$$G(x) \equiv \sum_{n=-\infty}^{+\infty} v_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx.$$

Then formally we have

$$\int_0^1 [F(x)u_n(x) + G(x)v_n(x)]dx \equiv \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx$$

or

$$\int_0^1 \{[F(x) - f(x)]u_n(x) + [G(x) - g(x)]v_n(x)\}dx = 0,$$
$$n = 0, \pm 1, \pm 2, \cdots.$$

If now we define

$$F(x) - f(x) \equiv \psi(x), \qquad G(x) - g(x) = -\varphi(x),$$

then by Theorem II we have

$$F(x) \equiv f(x), \qquad G(x) \equiv g(x), \qquad \text{q.e.d.}$$

THEOREM VII: *If $f(x)$, $g(x)$ do not satisfy (2) but possess continuous second derivatives as before, the series*

$$\begin{cases} c_0u_0(x) + [c_1u_1(x) + c_{-1}u_{-1}(x)] + [c_2u_2(x) + c_{-2}u_{-2}(x)] + \cdots, \\ c_0v_0(x) + [c_1v_1(x) + c_{-1}v_{-1}(x)] + [c_2v_2(x) + c_{-2}v_{-2}(x)] + \cdots, \end{cases} \tag{102}$$

converge uniformly to $f(x)$, $g(x)$ for $0 < \epsilon \leqq x \leqq 1 - \epsilon < 1$, where ϵ is an arbitrarily small positive number and (u_n, v_n), c_n are defined as before.

We have from (100)

$$c_nu_n(x) = \frac{u_n(x)}{\lambda_n}[g(x)u_n(x) - f(x)v_n(x)]_0^1 + O\left(\frac{1}{n^2}\right).$$

Hence by (91) for $|n|$ large

$$c_nu_n(x) = \frac{(-1)^nK_1u_n(x) - K_0u_n(x)}{n\pi} + O\left(\frac{1}{n^2}\right),$$

where

$$K_1 \equiv g(1)\sin P(1) - f(1)\cos P(1),$$
$$K_0 \equiv g(0)\sin P(0) - f(0)\cos P(0).$$

Again

$$c_{-n}u_{-n}(x) = \frac{(-1)^n K_1 u_{-n}(x) - K_0 u_{-n}(x)}{-n\pi} + O\left(\frac{1}{n^2}\right).$$

Hence

$$c_n u_n(x) + c_{-n}u_{-n}(x) = \left[\frac{(-1)^n K_1}{n\pi} - \frac{K_0}{n\pi}\right][u_n - u_{-n}].$$

But

$$u_n(x) - u_{-n}(x) = \sin[n\pi x + P(x)] - \sin[-n\pi x + P(x)] + O\left(\frac{1}{n}\right)$$

$$= 2\sin(n\pi x)\cos P(x) + O\left(\frac{1}{n}\right).$$

$\therefore\quad c_n u_n(x) + c_{-n}u_{-n}(x)$

$$= 2\cos P(x)\left[K_1(-1)^n\frac{\sin n\pi x}{n\pi} - K_0\frac{\sin n\pi x}{n\pi}\right] + O\left(\frac{1}{n^2}\right).$$

The series $\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi x}{n\pi}$, $\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n\pi}$ converge uniformly, for $0 < \epsilon \leqq x \leqq 1 - \epsilon < 1$, and since we get a similar expression for

$$c_n v_n(x) + c_{-n}v_{-n}(x),$$

the rest of the proof is like in the Theorem above.

Section V. Further Results on the Distribution of Principal Parameter Values.

Without the use of Theorem IV we can show that for the case in which

$$|C| < \sqrt{A^2 + B^2},$$

$D(\lambda)$ has exactly two roots in each interval

$$[(2k-1)\pi, (2k+1)\pi],\quad |k| \text{ large and integral,}$$

which have the asymptotic form (95) for $|n|$, $|\lambda|$ sufficiently large. Without loss of generality assume (58), then by definition $C = 2$. Then

$$\sqrt{A^2 + B^2} - 2 \equiv E > 0. \tag{103}$$

By (50)

$$D(\lambda) = \sqrt{A^2 + B^2}\sin[\xi(1) + \varphi] + C + \frac{K}{\lambda}, \qquad 0 \leqq |K| < M,$$

for $|\lambda|$ large, or

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = \sin\Lambda + \frac{C}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda},$$

where

$$\Lambda \equiv \xi(1) + \varphi. \tag{104}$$

For $\Lambda = (2k-1)\pi$ or $2k\pi$, and $|\lambda| > \lambda_M$,

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = \frac{2}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda} > 0.$$

For $\Lambda = (4k-1)\pi/2$, and $|\lambda| > \lambda_E$,

$$\frac{D(\lambda)}{\sqrt{A^2+B^2}} = -1 + \frac{2}{\sqrt{A^2+B^2}} + \frac{K'}{\lambda} = \frac{-E}{\sqrt{A^2+B^2}} + \frac{K'}{\lambda} < 0.$$

Let L_1 be the larger of λ_M, λ_E, then for $|\lambda| < L_1$, $D(\lambda)$ must have at least two roots in the intervals $(2k-1)\pi < \lambda < 2k\pi$. Since for $2k\pi \leqq \Lambda \leqq (2k+1)\pi$, $\sin\Lambda \geqq 0$, $D(\lambda)$ has no root there. To show that there are exactly two roots in the previous interval we use the fact that $D(\lambda) = 0$ only when

$$\sin\Lambda = -\frac{2}{\sqrt{A^2+B^2}} + \frac{K'}{\lambda}.$$

It is obvious that for $|\lambda| >$ some L_2 the right member is always negative since $|K'| < M$. If L is the larger of L_1, L_2, then, since

$$\sin\Lambda = \frac{E}{\sqrt{A^2+B^2}} - 1 + \frac{K'}{\lambda},$$

and when $|\lambda| > \lambda_E$

$$\frac{K'}{\lambda} < \frac{E}{\sqrt{A^2+B^2}},$$

we have

$$0 > \sin\Lambda > -1 \qquad |\lambda| > L.$$

Obviously there are just two values of Λ in any $[(2k-1)\pi,\ 2k\pi]$ interval which satisfy this inequality, i.e.,

$$\Lambda_1 = (2k-1)\pi + \psi + O\left(\frac{1}{\lambda}\right),$$

$$\Lambda_2 = 2k\pi - \psi + O\left(\frac{1}{\lambda}\right), \qquad 0 < \psi < \frac{\pi}{2},$$

where $\sin\psi \equiv \dfrac{-2}{\sqrt{A^2+B^2}}$.

$$\therefore\quad \Lambda = m\pi + (-1)^{m+1}\psi + O\left(\frac{1}{\lambda}\right),$$

for $|m|$ large, and by the definition of Λ

$$\lambda_n = m\pi + (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2}\,dx - \varphi + O\left(\frac{1}{m}\right),$$

when $|n|$ is large; or since in each 2π-interval there are 2 values of λ we have

$$\lambda_n = n\pi + s\pi + (-1)^{n+s+1}\psi - \int_0^1 \frac{a+b}{2}\,dx - \varphi + O\left(\frac{1}{n}\right),$$

s being a finite integer.

CASE of $C^2 = A^2 + B^2$: Here the above reasoning does not apply and we attack the problem by getting a more accurate asymptotic expansion

for $D(\lambda)$ and $D'(\lambda)$. If we assume

$$u(x) \equiv e^{\lambda\omega}\left[u_0 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \cdots\right],$$
$$v(x) \equiv e^{\lambda\omega}\left[v_0 + \frac{v_1}{\lambda} + \frac{v_2}{\lambda^2} + \cdots\right],$$

then substituting in (1) equate coefficients, we are led to the expansion

$$u(x) \equiv A_0 e^{\xi i} + B_0 e^{-\xi i} + \sum_{k=1}^{\infty} \frac{A_k e^{\xi i} + B_k e^{-\xi i}}{\lambda^k},$$
$$v(x) \equiv C_0 e^{\xi i} + D_0 e^{-\xi i} + \sum_{k=1}^{\infty} \frac{C_k e^{\xi i} + D_k e^{-\xi i}}{\lambda^k},$$

or since we restrict λ to real values, to

$$\left\{\begin{aligned} u(x) &\equiv A_0 \cos \xi + B_0 \sin \xi + \sum_{k=1}^{\infty} \frac{A_k \cos \xi + B_k \sin \xi}{\lambda^k}, \\ v(x) &\equiv C_0 \cos \xi + D_0 \sin \xi + \sum_{k=1}^{\infty} \frac{C_k \cos \xi + D_k \sin \xi}{\lambda^k}, \end{aligned}\right. \tag{105}$$

where ξ is defined as in (44) and A, B, C, D are independent of λ.

We may show that $(\alpha\gamma) = (\beta\delta)$, $(\alpha\delta) = (\gamma\beta)$ so that

$$A = 2(\alpha\gamma), \qquad B = 2(\alpha\delta), \qquad C = 2(\alpha\beta),$$

and since $(\alpha\beta) = 1$, we have

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = 1 + [u_2(1) - v_1(1)]\tfrac{1}{2} \cos \varphi + [u_1(1) + v_2(1)]\tfrac{1}{2} \sin \varphi,$$

in which (u_1, v_1), (u_2, v_2) are defined by (12) and φ by (52). From (50) we can show that $D'(\lambda)$ has a root in the interval

$$(2k - 1)\pi + \tfrac{1}{4}\pi < \Lambda < (2k - 1)\pi + \tfrac{3}{4}\pi. \tag{106}$$

By putting (105) in (1) and equating coefficients we get a set of equations from which we can solve for A, B, C, D and if we put that value of λ in for which $D'(\lambda) = 0$, we obtain

$$D(\lambda) - \frac{m_1 + 2m_0 m_1 \cos 2\varphi + m_0^2}{8\lambda^2} + O\left(\frac{1}{\lambda^3}\right),$$

wherein

$$m_1 \equiv \tfrac{1}{2}a(1) - \tfrac{1}{2}b(1), \qquad m_0 \equiv \tfrac{1}{2}a(0) - \tfrac{1}{2}b(0).$$

Hence $D(\lambda)$ will have two roots in the interval (106) unless

Case 1) $m_0 = m_1 = 0$;
Case 2) $\varphi \equiv \cos^{-1}(\alpha\gamma) \equiv \sin^{-1}(\alpha\delta) = \tfrac{1}{2}\pi$, $m_0 = m_1$;
Case 3) $\varphi = 0$, $m_0 = -m_1$.

By extending the calculation to the fifth set of functions A_4, B_4, C_4, D_4 we find that $D(\lambda)$ will still have two roots in (106) for Case 1) unless

$$m_1'^2 + 2m_0'm_1' \cos 2\varphi + m_0'^2 = 0,$$

i.e., unless

$$\begin{aligned} &1°. \quad m_0' = m_1' = 0; \\ &2°. \quad \varphi = \tfrac{1}{2}\pi, \quad m_0' = m_1'; \\ &3°. \quad \varphi = 0, \quad m_0' = -m_1'; \end{aligned}$$

in which

$$m_0' \equiv \tfrac{1}{2}a'(0) - \tfrac{1}{2}b'(0), \qquad m_1' \equiv \tfrac{1}{2}a'(1) - \tfrac{1}{2}b'(1).$$

The same is true for Case 2) unless

$$(m_1' - m_0')^2 + 4m_0^2(l_1 - l_0)^2 = 0$$

or

$$\begin{aligned} &1°. \quad m_1' = m_0', \quad m_0 = 0; \\ &2°. \quad m_1' = m_0', \quad l_1 = l_0; \end{aligned}$$

where

$$l_1 \equiv \tfrac{1}{2}a(1) + \tfrac{1}{2}b(1), \qquad l_0 \equiv \tfrac{1}{2}a(0) + \tfrac{1}{2}b(0).$$

It is probable that by this method $D(\lambda)$ would also be shown to have roots for Case 3) unless

$$(m_1' + m_0')^2 + 4m_0^2(l_1 + l_0)^2 = 0,$$

and that if one should continue the computation one would find that $D(\lambda)$ possesses roots unless

$$\begin{aligned} &\text{I.} \quad m_0 = m_1 = m_1' = m_0' = m_1'' = m_0'' \cdots = 0; \\ &\text{II.} \quad \varphi = \tfrac{1}{2}\pi, \quad m_0 = m_1, \quad m_0' = m_1', \quad m_0'' = m_1'', \quad \cdots; \\ &\text{III.} \quad \varphi = 0, \quad m_0 = -m_1, \quad m_0' = -m_1', \quad m_0'' = -m_1'', \quad \cdots. \end{aligned}$$

It is interesting to note that for a restricted case of I, namely

$$a(x) \equiv b(x) \equiv 0,$$

$D(\lambda)$ possesses only double roots, provided

$$\begin{cases} \alpha_1 = \gamma_1 = \beta_2 = \delta_2 = 1, \\ \alpha_2 = \gamma_2 = \beta_1 = \delta_1 = 0. \end{cases} \tag{107}$$

Here,

$$D(\lambda) = 2 + 2\cos\lambda,$$
$$\lambda = (2k - 1)\pi,$$

in fact

$$D(\lambda) \equiv \begin{vmatrix} 1 + \cos\lambda, & \sin\lambda \\ -\sin\lambda, & 1 + \cos\lambda \end{vmatrix}.$$

This leads to a special case of Fourier's Series in which the terms involving $\sin n\pi x$, $\cos n\pi x$ for n even are wanting.

Another interesting result is the following

THEOREM: *A necessary condition that the determinant*

$$D(\lambda) \equiv \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_1v_1) & U_2(u_2v_2) \end{vmatrix}$$

be of rank zero is that $(\alpha\beta) = (\gamma\delta)$, *where* (u_1, v_1), (u_2, v_2) *are defined by* (12).

We have

$$U_1(u_1v_1) \equiv \alpha_1 + \gamma_1 u_1(1) + \delta_1 v_1(1) = 0, \tag{108}$$

$$U_2(u_1v_1) \equiv \alpha_2 + \gamma_2 u_1(1) + \delta_2 v_1(1) = 0, \tag{109}$$

$$U_1(u_2v_2) \equiv \beta_1 + \gamma_1 u_2(1) + \delta_1 v_2(1) = 0, \tag{110}$$

$$U_2(u_2v_2) \equiv \beta_2 + \gamma_2 u_2(1) + \delta_2 v_2(1) = 0, \tag{111}$$

Multiply the first two equations by δ_2, $-\delta_1$ respectively, then

$$(\alpha\delta) + (\gamma\delta)u_1(1) = 0. \tag{112}$$

Similarly using $-\alpha_2$, α_1 on the last two we have

$$(\alpha\beta) + (\alpha\gamma)u_2(1) + (\alpha\delta)v_2(1) = 0. \tag{113}$$

Again using γ_2, $-\gamma_1$ to multiply the first two, we get

$$(\alpha\gamma) - (\gamma\delta)v_1(1) = 0. \tag{114}$$

Combining equations (112), (114), (113) we obtain

$$(\alpha\beta) - (\gamma\delta)\begin{vmatrix} u_1(1), & v_1(1) \\ u_2(1), & v_2(1) \end{vmatrix} = 0,$$

or

$$(\alpha\beta) = (\gamma\delta). \qquad \text{Q.e.d.}$$

CONFORMAL TRANSFORMATIONS OF PERIOD n AND GROUPS GENERATED BY THEM.*

By Harry Langman.

Introduction.

Professor Kasner† has discussed the characteristics of groups of transformations generated by conformal transformations of period 2. He considered both the "direct" and what he terms "reverse," or "improper" transformations. The former type may be represented in the form $Z = f(z)$, where $f(z)$ is analytic at the origin, and converts it into itself. The latter type, termed a "symmetry," may be represented in the form $Z = f(z_0)$, where z_0 is the conjugate of z.

In discussing direct transformations, Professor Kasner utilizes the implicit form

$$Z + z = d_2(Z - z)^2 + d_4(Z - z)^4 + \cdots,$$

which includes every transformation of period 2, and every solution of which, for arbitrary values of the coefficients d, yields a transformation of period 2. In considering the conditions under which a given transformation

$$Z = c_1z + c_2z^2 + c_3z^3 + \cdots$$

can be factored as the product of two transformations of period 2, Kasner obtains the condition $c_3 - c_2^2 = 0$, besides the obvious condition $c_1 = 1$. If $c_2 = 0$, further conditions must be satisfied.

In the case of reverse transformations, the given transformation can always be factored into two "symmetries" if $|c_1| = 1$, and the angle of c_1 is incommensurable with π. If the latter condition is not satisfied, then the given transformation can in any case be factored into four symmetries.

It is the purpose of this paper to generalize the results obtained by Kasner to include transformations of period n. In the case of direct transformations, it is remarkable that no such necessary condition as that obtained by Kasner upon the coefficients following the first is found necessary for factorization into transformations of periods greater than 2. In the case $c_1 = 1$ there is, however, a non-zero relation between the coefficients immediately following the first in the given transformation and the period of the factor transformations (the periods of the latter then being neces-

* Presented to the American Mathematical Society December 28, 1920.

† "Infinite Groups Generated by Conformal Transformations of Period 2 (Involutions and Symmetries)," American Journal of Mathematics, XXXVIII, 2, 1916.

sarily equal). In any case, this condition can be obviated by either changing the period of the factor transformations or factoring into three transformations of given period. The complete result is given in Theorem VI.

In the case of reverse conformal transformations, it will be shown that all such transformations are of irreducible period 2; therefore no others exist than those discussed by Kasner.

The parametric form utilized for transformations of regular period is a general solution of Babbage's equation,* and the reduction to the form $f = \varphi^{-1}\epsilon\varphi$ yields $\varphi f = \epsilon\varphi$, a special case of Schröder's equation.†

In the following discussion, the terms function and transformation are used interchangeably, and questions of convergence are not gone into, the analysis being purely formal.

PRELIMINARY DISCUSSION; PARAMETRIC FORM OF PERIODIC TRANSFORMATIONS.

1. Let $f(z)$ be defined by

$$f(z) = \lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \tag{1}$$

where $\lambda_1 \neq 0$. We may introduce the notation

$$f_1(z) = f(z), \qquad f_{s+1}(z) = f[f_s(z)]; \qquad s = 1, 2, 3, \cdots. \tag{2}$$

Assuming the transformation defined by (1) to be of period n, we have

$$f_n(z) = z, \qquad f_{kn+s}(z) = f_s(z); \qquad k = 1, 2, 3, \cdots. \tag{3}$$

We have obviously

$$\lambda_1^n = 1. \tag{4}$$

If in (1) we choose $\lambda_1 = 1$, we have

$$f_n(z) = z + n\lambda_2 z^2 + \cdots;$$

hence we must have $\lambda_2 = 0$. Similarly, all the other coefficients vanish. Hence we have

THEOREM I. *If the transformation*

$$\lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots$$

be of period n, we must have $\lambda_1^n = 1$; if $\lambda_1 = 1$, the transformation reduces to identity.

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* S. Pincherle: "Functional Equations and Operations," *Encyklopädie d. Math. Wiss.*, II, A 11, and *Encyclopedie d. Sci. Math.*, II, 26. Also O. Rausenberger, "Lehrbuch der Theorie der periodischen Funktionen," Leipzig, 1884, p. 162; A. A. Bennett, "The Iteration of Functions of One Variable," *Annals of Math.*, 2d series, 17 (1915).

† Ibid. In our case ϵ is taken so that $\epsilon^n = 1$. G. A. Pfeiffer, *Trans. Am. Math. Soc.*, XVIII, 2, pp. 185–198, considers the complementary case, $|\epsilon| = 1$ with incommensurable angle, and discusses the convergence and divergence of the solutions obtained.

2. Assuming (1) to be of period n, we may write it in the form

$$f(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots; \qquad \epsilon^n = 1. \tag{5}$$

Suppose now ϵ *not* a primitive root of $\epsilon^n = 1$. Suppose ϵ a primitive root of $\epsilon^m = 1$. Then $rm = n$, where $r > 1$. Denoting $f_m(z)$ by $g(z)$, we have

$$g_r(z) = f_{rm}(z) = z.$$

We have obviously $g(z) = \epsilon^m z + \cdots$; hence, by Theorem I,

$$g(z) = f_m(z) = z.$$

Hence the transformation (5) is of period m. We have then

THEOREM II. *If a periodic transformation be expressed in the form*

$$\epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots,$$

where

$$\epsilon^m = 1,$$

*then the transformation is of period m.**

Hence in the form (5) we may conveniently restrict ourselves to the case where ϵ is a primitive root of $\epsilon^n = 1$. In the following discussion, we shall presume this to be the case.

3. Suppose we consider the n variables $z_1, z_2, \cdots, z_n$, between which we have the $n - 1$ relations

$$z_{t+1} = f(z_t); \qquad t = 1, 2, \cdots, n-1, \tag{6}$$

where

$$f(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \tag{7}$$

ϵ being a primitive nth root of unity. We may introduce the linear substitution

$$z_t = \epsilon^t x_1 + \epsilon^{2t} x_2 + \cdots + \epsilon^{st} x_s + \cdots + \epsilon^{nt} x_n; \quad t = 1, 2, \cdots, n. \tag{8}$$

This is reversible; we have

$$x_s = \frac{1}{n}\left(\epsilon^{-s} z_1 + \epsilon^{-2s} z_2 + \cdots + \epsilon^{-ts} z_t + \cdots + \epsilon^{-ns} z_n\right); \quad s = 1, 2, \cdots, n. \tag{9}$$

By means of (6) and (7), each variable z can be expressed formally as a power series in z_1. We have

$$z_t = \epsilon^{t-1} z_1 + \cdots; \qquad t = 2, 3, \cdots, n. \tag{10}$$

* If (5) take the form

$$f(z) = \epsilon z + \lambda_r z^r + \cdots,$$

we obtain

$$f_n(z) = \epsilon^n z + \epsilon^{n-1}\lambda_r[1 + \epsilon^{r-1} + \epsilon^{2(r-1)} + \cdots + \epsilon^{(n-1)(r-1)}]z^r + \cdots.$$

Hence if $r - 1 \equiv 0 \pmod{n}$, we must have $\lambda_r = 0$. Hence the next coefficient to appear after the first cannot be of order $kn + 1$.

From (9), then,

$$x_1 = \epsilon^{-1} z_1 + \cdots. \tag{11}$$

From (11), we have z_1 expressible as a power series in x_1. Hence from (10) each z is so expressible. From (9), then, each x can be expanded formally as a power series in x_1 with coefficients uniquely determined in terms of those of f in (7). We may then write

$$x_s = A_{s,1} x_1 + A_{s,2} x_1^2 + \cdots + A_{s,r} x_1^r + \cdots; \quad s = 2, 3, \cdots, n. \tag{12}$$

Hence the $n - 1$ relations (6) among the variables z may be replaced by the $n - 1$ implicit relations (12) where each variable x is expressed in terms of the variables z in the form (9). If z_2 is defined in terms of z_1 in the form (7), then we may introduce the additional variables $z_3, z_4, \cdots, z_n$, similarly defined; hence z_2 may be defined in terms of z_1 in the implicit form (12), involving the elimination of the additional variables $z_3, z_4, \cdots, z_n$.

4. Suppose now the transformation defined by (7) to be of period n. Then, since $z_t = f_{t-1}(z_1)$,

$$z_{n+1} = f_n(z_1) = z_1 = f(z_n). \tag{13}$$

Using the substitution (8), we have then the following n relations between the variables x:

$$\begin{aligned}
&\epsilon^2 x_1 + \cdots + \epsilon^{2s} x_s + \cdots + \epsilon^{2n} x_n \\
&\qquad\qquad = f(\epsilon x_1 + \cdots + \epsilon^s x_s + \cdots + \epsilon^n x_n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
&\epsilon^{t+1} x_1 + \cdots + \epsilon^{s(t+1)} x_s + \cdots + \epsilon^{n(t+1)} x_n \\
&\qquad\qquad = f(\epsilon^t x_1 + \cdots + \epsilon^{st} x_s + \cdots + \epsilon^{nt} x_n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
&\epsilon^n x_1 + \cdots + \epsilon^{sn} x_s + \cdots + \epsilon^{nn} x_n \\
&\qquad\qquad = f(\epsilon^{n-1} x_1 + \cdots + \epsilon^{s(n-1)} x_s + \cdots + \epsilon^{n(n-1)} x_n), \\
&\epsilon x_1 + \cdots + \epsilon^s x_s + \cdots + \epsilon^n x_n \\
&\qquad\qquad = f(\epsilon^n x_1 + \cdots + \epsilon^{sn} x_s + \cdots + \epsilon^{nn} x_n).
\end{aligned} \tag{14}$$

From the previous discussion, we note that the first $n - 1$ of equations (14) will yield unique expansions in power series of x_1 in the form (12) for the variables $x_2, x_3, \cdots, x_n$. We may represent the relations (12) in the following notation:

$$x_s = \theta_s(x_1); \qquad s = 2, 3, \cdots, n. \tag{15}$$

If we now introduce in (14) the substitution

$$x_s = \epsilon^s y_s; \qquad s = 1, 2, \cdots, n, \tag{16}$$

equations (15) become

$$y_s = \epsilon^{-s} \theta_s(\epsilon y_1); \qquad s = 2, 3, \cdots, n. \tag{17}$$

We should observe, however, that the last $n-1$ equations of (14) also yield definite power series in x_1 for the other variables x and that these must be identical with the relations (15). But on introducing the quantities y in the first $n-1$ equations of (14), in the form (16), we observe that the relations between the variables y are identical with those between the corresponding variables x in the last $n-1$ equations of (14). Hence the relations between the variables y must be the same as those between the corresponding variables x. Hence we have

$$y_s = \theta_s(y_1); \qquad s = 2, 3, \cdots, n. \tag{18}$$

These of course must be consistent with (17). Hence

$$\theta_s(y_1) = \epsilon^{-s}\theta_s(\epsilon y_1); \qquad s = 2, 3, \cdots, n. \tag{19}$$

Replacing y_1 in (19) successively by ϵy_1, we obtain, on dropping the subscript,

$$\theta_s(y) = \epsilon^{-ts}\theta_s(\epsilon^t y); \qquad s = 2, 3, \cdots; \qquad t = 1, 2, 3, \cdots. \tag{20}$$

From the form (12), (20) may be written

$$\theta_s(y) = \epsilon^{-ts}\sum_{r=1}^{\infty} \epsilon^{tr} A_{s,\,r}\cdot y^r; \quad s = 2, 3, \cdots, n; \quad t = 1, 2, 3, \cdots. \tag{21}$$

Letting t assume the range of values* 1 to n in (21), and adding the resulting series, we obtain the form

$$\theta_s(y) = A_{s,\,s}y^s + A_{s,\,s+n}y^{s+n} + A_{s,\,s+2n}y^{s+2n} + \cdots; \quad s = 2, 3, \cdots, n. \tag{22}$$

From the form of the series (22) we observe that the double subscripts are not now necessary. We may then write the series (15) in the form

$$\begin{aligned} x_2 &= a_2x_1^2 + a_{n+2}x_1^{n+2} + a_{2n+2}x_1^{2n+2} + \cdots, \\ x_3 &= a_3x_1^3 + a_{n+3}x_1^{n+3} + a_{2n+3}x_1^{2n+3} + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ x_s &= a_sx_1^s + a_{n+s}x_1^{n+s} + a_{2n+s}x_1^{2n+s} + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ x_n &= a_nx_1^n + a_{2n}x_1^{2n} + a_{3n}x_1^{3n} + \cdots. \end{aligned} \tag{23}$$

Replacing the variables x in (23)† by the corresponding expressions in the variables z from (9), we have then $n-1$ implicit relations among those variables in place of the $n-1$ direct relations (6).

5. If we now write

$$\frac{1}{n}(\epsilon^{-1}z_1 + \epsilon^{-2}z_2 + \cdots + \epsilon^{-t}z_t + \cdots + \epsilon^{-n}z_n) = r, \tag{24}$$

* This is equivalent to permuting the substitution (16) in equations (14).

† These equations have already been obtained by Bennett, loc. cit.

equations (23) may be written

$$\begin{aligned}
&\frac{1}{n}(\epsilon^{-2}z_1+\epsilon^{-4}z_2+\cdots+\epsilon^{-2t}z_t+\cdots+\epsilon^{-2n}z_n)=a_2r^2+a_{n+2}r^{n+2}+\cdots,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot,\\
&\frac{1}{n}(\epsilon^{-s}z_1+\epsilon^{-2s}z_2+\cdots+\epsilon^{-ts}z_t+\cdots+\epsilon^{-ns}z_n)=a_sr^s+a_{n+s}r^{n+s}+\cdots,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot,\\
&\frac{1}{n}(\epsilon^{-n}z_1+\epsilon^{-2n}z_2+\cdots+\epsilon^{-tn}z_t+\cdots+\epsilon^{-nn}z_n)=a_nr^n+a_{2n}r^{2n}+\cdots.
\end{aligned}\tag{25}$$

Multiplying equations (24) and (25) seriatim by ϵ^t, ϵ^{2t}, $\cdots$, ϵ^{st}, $\cdots$, ϵ^{nt}, and adding, we obtain

$$\begin{aligned}
z_t = \epsilon^t r &+ \epsilon^{2t}(a_2r^2 + a_{n+2}r^{n+2} + \cdots)\\
&+\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+ \epsilon^{st}(a_sr^s + a_{n+s}r^{n+s} + \cdots)\\
&+\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+ \epsilon^{nt}(a_nr^n + a_{2n}r^{2n} + \cdots),
\end{aligned}\tag{26}$$

for all values of t from 1 to n. If we now define the function φ by

$$\varphi(z) = z + \sum_{k=0}^{\infty}\sum_{r=1}^{n-1} a_{kn+r+1}z^{kn+r+1},\tag{27}$$

equations (26) take the form

$$z_t = \varphi(\epsilon^t r); \qquad t = 1, 2, \cdots, n.\tag{28}$$

In particular,

$$z_1 = \varphi(\epsilon r), \qquad z_2 = \varphi(\epsilon^2 r),\tag{29}$$

which are equivalent to

$$z_1 = \varphi(r), \qquad z_2 = \varphi(\epsilon r),\tag{29'}$$

defining z_2 as a function of z_1 in terms of the parameter r. Furthermore, every pair of equations (28) determines either z uniquely as an integral power series in the other with coefficients expressible as polynomials in the coefficients a in (27) of the same and lower orders. This is obviously also true if instead of φ as defined by (27) we use the more general form

$$\psi(r) = r + \sum_{t=2}^{\infty} a_t r^t.\tag{30}$$

6. From the form of equations (28) we observe that if for an arbitrary set of coefficients a we obtain $z_2 = f(z_1)$, we must also have $z_3 = f(z_2)$, etc. Hence $z_{n+1} = f(z_n)$. But $z_1 = z_{n+1} = z_{2n+1}$, etc. Hence $z_1 = f(z_n)$, from which $f_n(z) = z$. Hence $f(z)$ is of period n. Furthermore, if z_α and z_β be any two of the z's defined by (28), either is expressible as a periodic

function of the other. If $|\alpha - \beta|$ is prime to n, the order is also n. If only equations (29) are given, or the corresponding equations with ψ instead of φ, the remaining z's may be introduced in the form (28), yielding z_2 as a function of z_1 of period n. Hence we have

THEOREM III. *Every transformation of period n, $y = f(z)$, can be put uniquely into the form*

$$z = \varphi(r), \qquad y = \varphi(\epsilon r); \qquad \epsilon^n = 1,$$

where

$$\varphi(r) = r + \sum_{k=0}^{\infty} \sum_{t=1}^{n-1} a_{kn+t+1} r^{kn+t+1};$$

and every solution of these equations for arbitrary values of the coefficients defines $y = f(z)$ uniquely as a transformation of period n.

*In other words, a necessary and sufficient condition for $y = f(z)$ to be a transformation of period n is that it be a solution of equations of the form $z = \varphi(r)$, $y = \varphi(\epsilon r)$.**

Hence the two forms are equivalent, and we may conveniently confine our attention to the inclusive form (29′).

7. If the transformation (5) result from the elimination of r between the equations (29′), then we obtain

$$\lambda_t = (\epsilon^t - \epsilon)a_t + (\text{polynomial in } \lambda\text{'s and } a\text{'s of order} < t),$$

which can obviously be put into the form

$$\lambda_t = (\epsilon^t - \epsilon)a_t + (\text{polynomial in } a\text{'s of order} < t). \qquad (31)$$

The quantities a in (31) are arbitrary. For each new λ_t, where t is not of the form $kn + 1$, a new arbitrary a_t is introduced. Furthermore, each a can be expressed as a polynomial in λ's of corresponding and lower orders. Hence the λ's in (5) of orders not of the form $kn + 1$ can be taken arbitrarily, the remaining λ's being then determined necessarily in the form

$$\lambda_{kn+1} = R_k(\lambda_{kn}, \lambda_{kn-1}, \cdots); \qquad k = 1, 2, \cdots, \qquad (32)$$

where the R's are rational integral functions. Hence we have

THEOREM IV. *If*

$$y = \epsilon z + \lambda_2 z^2 + \cdots; \qquad \epsilon^n = 1$$

be a transformation of period n, all coefficients of orders not of the form $kn + 1$

* If $n = 2$, $\epsilon = -1$, and equations (27) and (29′) yield $z_1 = r + a_2 r^2 + a_4 r^4 + \cdots$, $z_2 = -r + a_2 r^2 + a_4 r^4 + \cdots$, from which

$$\frac{z_2 + z_1}{2} = a_2 \left(\frac{z_2 - z_1}{2}\right)^2 + a_4 \left(\frac{z_2 - z_1}{2}\right)^4 + \cdots,$$

the form utilized by Kasner.

may be taken arbitrarily, the remaining coefficients being determined as necessary rational integral functions of the coefficients of lower order.

In other words, $k(n-1)$ out of the kn coefficients following the first may be taken arbitrarily.

The relations (32) are then necessary and sufficient for the transformation (5) to be of period n. If now, instead of (29′), we define the relation $y = f(z)$ in the form

$$z = \psi(r), \qquad y = \psi(\epsilon r), \tag{33}$$

where ψ has the form (30), then f is also of period n. In other words, (33) is no more general than (29′). In this case too the relations (31) will not yield expressions in λ's for the a's of order $kn+1$. On the other hand, the λ's of order $kn+1$ will be expressed in terms of a's of order $k'n+1$, $k' < k$, as well as the other a's. But since the relations (32) between the λ's are *necessary*, it follows that the a's of order $k'n+1$ will be automatically eliminated on eliminating the remaining a's from the expressions for λ_{kn+1} and the λ's of lower order.

Hence, as already observed, any two equations $z_\alpha = \varphi(\epsilon^\alpha r)$ and $z_\beta = \varphi(\epsilon^\beta r)$, of (28), or the corresponding equations with ψ instead of φ, determine either z as a periodic function of the other. The period is the quotient between n and the greatest common divisor of n and $|\alpha - \beta|$.

8. If the function

$$f(z) = \lambda_1 z + \lambda_2 z^2 + \cdots \tag{34}$$

result from the relation

$$f(z) = g^{-1}[\epsilon g(z)]; \qquad \epsilon^n = 1, \tag{35}$$

where

$$g(z) = A_1 z + A_2 z^2 + \cdots; \qquad A_1 \neq 0, \tag{36}$$

then $f(z)$ is obviously of period n. We shall now show that every periodic transformation $f(z)$ can be expressed in the form (35).*

Suppose f is given where, symbolically, $f^n = 1$. We inquire now whether g can always be found so as to satisfy (35). This condition is equivalent to

$$g[f(z)] = \epsilon g(z). \tag{37}$$

It is to be observed that, whether we obtain f when g is given, or g when f is given, the *same* set of formal identities obtained by equating the coefficients of like powers of z in (37) must be considered. These take the form

$$\begin{aligned} &A_1\lambda_1 = \epsilon A_1, \qquad A_1\lambda_2 + A_2\epsilon^2 = \epsilon A_2, \qquad \cdots, \\ &A_1\lambda_t + 2A_2\epsilon\lambda_{t-1} + \cdots + A_t\epsilon^t = \epsilon A_t, \qquad \cdots. \end{aligned} \tag{38}$$

We observe that A_1 may be taken arbitrarily subject, of course, to the

* Cf. S. Pincherle, O. Rausenberger and A. A. Bennett, loc. cit.

condition $A_1 \neq 0$. We have $\lambda_1 = \epsilon$. Furthermore, each coefficient A_t, $t < n+1$, is expressed as A_1 (polynomial in λ's). Hence we have λ_{n+1} expressed as a necessary rational integral function of λ's of lower order. Similarly, λ's of order $kn+1$ are expressed as rational integral functions of lower λ's and of arbitrary A's of order $k'n+1$, $k' < k$. But since every solution $f(z)$ where $g(z)$ in (35) is given is of period n, we have then a set of *necessary* conditions for the coefficients λ of order $kn+1$ of the form (32). Hence on eliminating the coefficients A of orders other than those of the form $kn+1$ from the expressions for λ_{kn+1} and λ's of lower order, the coefficients $A_{k'n+1}$ $(k' < k)$ must be eliminated automatically, the resulting conditions being identical with those represented by (32).

Furthermore, if (34) is given, we can find a function $g(z)$ consistent with (35) only if the same set of conditions (32) are satisfied between the coefficients λ. But as just seen these conditions are none other than those necessary for $f(z)$ to be a periodic transformation of order n. Hence every transformation (34) of order n can be expressed in the form (35). Furthermore, given g, f results uniquely, but not vice versa, since the coefficients of form A_{kn+1} may be taken arbitrarily, subject to the single restriction $A_1 \neq 0$. Hence we may conveniently choose

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, \cdots,$$

from which g takes the form

$$g(z) = z + \sum_{k=0}^{\infty} \sum_{r=1}^{n-1} A_{kn+r+1} z^{kn+r+1}, \tag{39}$$

in which case, from the form of equations (38), each coefficient A is determined uniquely in terms of the coefficients λ of the given transformation (34). Hence we have

THEOREM V. *Every transformation of the form*

$$f(z) = g^{-1}[\epsilon g(z)], \qquad \epsilon^n = 1,$$

where

$$g(z) = A_1 z + A_2 z^2 + \cdots; \qquad A_1 \neq 0,$$

is of period n, and every transformation $f(z)$ of period n may be expressed in the form $g^{-1}[\epsilon g(z)]$; furthermore, if we choose

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, \cdots,$$

the function $g(z)$ corresponds uniquely to $f(z)$.

In other words, every periodic transformation is conformally reducible to the form $Y = \epsilon Z$, a rotation about the origin through the angle $2\pi/n$.*

It is interesting to compare the functions φ and g in (27) and (39). Obviously each corresponds uniquely to the other and either may be used in place of the other or its inverse in the above equations.

* A simple example of a periodic transformation is obtained by taking $g(z) = z/(z-1)$, from which $f(z) = \epsilon z/[(\epsilon - 1)z + 1]$.

9. The method followed in the preceding discussion will enable us to write down readily the general solutions of some special additional functional equations. Suppose f given, where $f^n = 1$, and it is required to find g where, symbolically, $g^m = f$.* We may anticipate the number of arbitrary coefficients involved in the general solution. In that for $g^{mn} = 1$, out of tmn coefficients following the first a proportion of $(mn - 1)/mn$ may be taken arbitrary. For $f^n = 1$, the proportion $(n - 1)/n$ are arbitrary. Having assigned f, the proportion still remaining arbitrary is

$$\frac{mn - 1}{mn} - \frac{n - 1}{n} = \frac{m - 1}{mn}.$$

Suppose $f = \varphi^{-1}\epsilon\varphi$, where $\epsilon^n = 1$, and suppose h the general solution of $h^m = \epsilon$. Then $g = \varphi^{-1}h\varphi$ is the general solution of $g^m = f$. We have then to consider the functional equation $h^m = \epsilon$.

Let $\omega^m = \epsilon$; then $\omega^{mn} = 1$. Putting mn for n and ω for ϵ in (3) we observe that $z_{km+s} = \epsilon^k z_s$ for all values of k and s. Equation (24) becomes

$$\frac{1}{m}(\omega^{-1}z_1 + \omega^{-2}z_2 + \cdots + \omega^{-m}z_n) = r.$$

For all values of $s \not\equiv 1 \pmod{n}$, we have from (25)

$$\frac{1}{m}(\omega^{-s}z_1 + \omega^{-2s}z_2 + \cdots + \omega^{-ms}z_m) = a_s r^s + a_{n+s}r^{n+s} + \cdots;$$

for all other values of s the left members of (25) vanish. Applying the method of Section 5, we have

$$z_t = \varphi_1(\omega^t r),$$

where

$$\varphi_1(r) = r + \sum_{p=0}^{\infty}\sum_{k=1}^{m-1} a_{(pm+k)n+1}r^{(pm+k)n+1}. \tag{40}$$

Hence h is defined by

$$z = \varphi_1(r), \qquad h(z) = \varphi_1(\omega r), \qquad \omega^m = \epsilon, \qquad \epsilon^n = 1.$$

Hence the most general solution for $g^m = f$ where $f^n = 1$ is given by

$$g = \varphi^{-1}h\varphi = \varphi^{-1}\varphi_1\omega\varphi_1^{-1}\varphi.\dagger$$

* This is a very special case of the more general problem, the solution of which is equivalent to finding an n-section of a curvilinear angle in the sense employed by Kasner as an extension of the idea of symmetry and corresponding bisection due to Schwarz. See G. A. Pfeiffer, "On the Conformal Geometry of Analytic Arcs," AMER. JOUR. MATH., XXXVII, 4 (1915).

† A less specific form may be obtained more directly. Suppose g in form $\lambda^{-1}\omega\lambda$. Then $\lambda^{-1}\epsilon\lambda = f = \varphi^{-1}\epsilon\varphi$, whence $\varphi\lambda^{-1}\epsilon = \epsilon\varphi\lambda^{-1}$. Putting μ for $\varphi\lambda^{-1}$ we have $\mu(\epsilon z) = \epsilon\mu(z)$; hence μ must be of the form $\mu(z) = z\theta(z^n)$. Hence if φ is given we may choose $\lambda = \mu^{-1}\varphi$ in $g = \lambda^{-1}\omega\lambda = \varphi^{-1}\mu\omega\mu^{-1}\varphi$ as the most general solution of $g^m = f$. By Theorem V it is evident that there is a sufficient number of arbitrary constants, though not indicated in the very explicit form (40).

We may similarly write down the general solution of the functional equation

$$z + f(z) + f_2(z) + \cdots + f_{n-1}(z) = 0.$$

We observe that if $1 + f + \cdots + f^{n-1} = 0$, then $f^n - 1 = 0$ and f is a periodic function. The converse is not necessarily the case, as is readily verified in the special case

$$f(z) = \epsilon z + \epsilon z^2 + \epsilon z^3 + \cdots; \qquad \epsilon^3 = 1,$$

which satisfies $f^3 = 1$ but not $1 + f + f^2 = 0$.

We note that the left member of the last of equations (25) vanishes according to this condition. The general solution is then at once written in the form

$$z = \varphi_2(r), \qquad f(z) = \varphi_2(\epsilon r),$$

where

$$\varphi_2(z) = z + \sum_{k=0}^{\infty} \sum_{s=1}^{n-2} a_{kn+s+1} z^{kn+s+1}. \tag{41}$$

Similarly, the general solutions are obtained for equations formed by equating any of the left members of (25) to zero. A similar result may be obtained when the condition (yielding $n - 1$ similar conditions) is a linear relation obtained by eliminating $s - 1$ z's after equating s of these expressions to zero.

It may be shown that, corresponding to every periodic function f, where $f_n = 1$, there are others F, for which, symbolically,

$$1 + F + F^2 + \cdots + F^{n-1} = 0.*$$

---

* Instead of the substitution (8), we may effect the substitution $z_k = \sum_{t=1}^{n} \alpha_{k,\,t} u_t$ $(k = 1, 2, \cdots, n)$, where the α's form a set of orthogonal numbers. We shall for convenience write $\alpha_{p,\,n+q} = \alpha_{p+n,\,q} = \alpha_{p,\,q}$, and choose $\alpha_{k,\,n} = 1/\sqrt{n}$ $(k = 1, 2, \cdots, n)$. The last requires $\sum_{s=1}^{n} \alpha_{s,\,r} = 0$ $(r = 1, 2, \cdots, n - 1)$.

We have then n equations in n u's. Each $n - 1$ of these determine all u's in terms of u_1: $u_k = \theta_k(u_1)$, $k = 2, 3, \cdots, n$. Further, the functions so determined are unique. We now introduce the further substitution $u_k = \sum_{t=1}^{n-1} \beta_{k,\,t} v_t$, $u_n = v_n$ $(k = 1, 2, \cdots, n - 1)$, where $\beta_{s,\,t} = \sum_{p=1}^{n} \alpha_{p,\,s} \alpha_{p+1,\,t}$. It is readily verified that the $(n - 1)^2$ quantities β form a set of orthogonal numbers.

We now have

$$\begin{aligned}
z_k = \sum_{t=1}^{n} \alpha_{k,\,t} u_t &= \alpha_{k,\,n} v_n + \sum_{t=1}^{n-1} \left(\alpha_{k,\,t} \sum_{s=1}^{n-1} \beta_{t,\,s} v_s \right) \\
&= \alpha_{k+1,\,n} v_n + \sum_{s=1}^{n-1} \left(v_s \sum_{t=1}^{n-1} \alpha_{k,\,t} \beta_{t,\,s} \right) \\
&= \alpha_{k+1,\,n} v_n + \sum_{s=1}^{n-1} \left[v_s \sum_{p=1}^{n} \left(\alpha_{p+1,\,s} \sum_{t=1}^{n-1} \alpha_{k,\,t} \alpha_{p,\,t} \right) \right] \\
&= \sum_{s=1}^{n-1} \left[v_s \left(-\frac{1}{n} \sum_{p=1}^{n} \alpha_{p+1,\,s} + \alpha_{k+1,\,s} \right) \right] + \alpha_{k+1,\,n} v_n \\
&= \sum_{s=1}^{n} \alpha_{k+1,\,s} v_s.
\end{aligned}$$

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10. Kasner has considered the question whether any given transformation may be factored into two transformations each of period 2. In other

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Hence z_k is the same function of the v's as z_{k+1} is of the corresponding u's. But the latter are uniquely determined by any set of $n-1$ equations. Hence the relations among the v's are the same as those among the corresponding u's. Hence the θ-functions remain unchanged under transformations of the form $u_k \to \sum_{t=1}^{n-1} \beta_{k,\,t} u_t,\ u_n \to u_n\ (k = 1, 2, \cdots, n-1)$.

Putting $F(u_1) = \sum_{k=1}^{n-1} \beta_{1,\,k}\theta_k(u_1)$, we have $\theta_k[F(u_1)] = \sum_{p=1}^{n-1} \beta_{k,\,p}\theta_p(u_1)$. We have then

$$
\begin{aligned}
F_2(u_1) &= \sum_{k=1}^{n-1}\left[\beta_{1,\,k}\sum_{p=1}^{n-1}\beta_{k,\,p}\theta_p(u_1)\right] = \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{k=1}^{n-1}\beta_{1,\,k}\beta_{k,\,p}\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{k=1}^{n-1}\left\{\left(\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+1,\,k}\right)\left(\sum_{t=1}^{n}\alpha_{t,\,k}\alpha_{t+1,\,p}\right)\right\}\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s,t=1}^{n}\left(\alpha_{s,\,1}\alpha_{t+1,\,p}\sum_{k=1}^{n-1}\alpha_{s+1,\,k}\alpha_{t,\,k}\right)\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\left(\alpha_{s,\,1}\alpha_{s+2,\,p}\sum_{k=1}^{n-1}\alpha^2_{s+1,\,k}\right)+\sum_{\substack{s,t=1\\ s+1\neq t}}^{n}\left(\alpha_{s,\,1}\alpha_{t+1,\,p}\sum_{k=1}^{n-1}\alpha_{s+1,\,k}\alpha_{t,\,k}\right)\right\}\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left(\frac{n-1}{n}\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p}-\frac{1}{n}\sum_{\substack{s,t=1\\ s+1\neq t}}^{n}\alpha_{s,\,1}\alpha_{t+1,\,p}\right)\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p}-\frac{1}{n}\left(\sum_{s=1}^{n}\alpha_{s,\,1}\right)\left(\sum_{t=1}^{n}\alpha_{t+1,\,p}\right)\right\}\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p}\right] = \sum_{p=1}^{n-1}a_{p,2}\,\theta_p(u_1)
\end{aligned}
$$

if we introduce $a_{p,\,q} = \sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+q,\,p}$.

Similarly, we obtain

$$F_m(u_1) = \sum_{p=1}^{n-1} a_{p,\,m}\theta_p(u_1).$$

We have then

$$
\begin{aligned}
u_1+\sum_{p=1}^{n-1}F_p(u_1) &= u_1+\sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n-1}a_{p,\,s}\right] = u_1+\sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n-1}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,p}\right)\right]\\
&= u_1+u_1\sum_{s=1}^{n-1}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,1}\right)\\
&\qquad\qquad+\sum_{p=2}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,p}\right)-\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t,\,p}\right\}\right]\\
&= u_1-u_1\sum_{t=1}^{n}\alpha^2_{t,\,1}=0.
\end{aligned}
$$

The last result also follows from the relation $F_m(u_1) = \sum_{t=1}^{n}\alpha_{t,\,1}z_{t+m}$, readily deduced. In a similar way, we may also obtain

$$\theta_k^{(m)}(u_1) = \sum_{t=1}^{n-1}\left(u_t\sum_{p=1}^{n}\alpha_{p,\,k}\alpha_{p+m,\,t}\right),$$

where

$$\theta_k^{(m)}(u_1) = \theta_k^{(m-1)}[F(u_1)], \qquad \theta'_k(u_1) = \theta_k(u_1),$$

yielding

$$\sum_{m=1}^{n}\theta_k^{(m)}(u_1) = 0,$$

from which the previous result also follows.

words, given F, can f and g be found such that, symbolically,

$$F = gf; \qquad f^2 = 1, \qquad g^2 = 1?$$

We may generalize this inquiry as follows: Given $F(z)$, can $f(z)$ and $g(z)$ be obtained where, with the notation (2),

$$F(z) = g[f(z)]; \qquad f_m(z) = z, \qquad g_n(z) = z? \tag{42}$$

We shall assume these functions to be defined as follows:

$$\begin{aligned} F(z) &= K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \\ f(z) &= \epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \qquad \epsilon_1^m = 1; \\ g(z) &= \epsilon_2 z + \mu_2 z^2 + \mu_3 z^3 + \cdots, \qquad \epsilon_2^n = 1. \end{aligned} \tag{43}$$

We shall assume here that ϵ_1 and ϵ_2 are primitive roots of unity of orders m and n, respectively. It is to be observed that all the coefficients λ and μ in (43) may be taken arbitrarily, consistent with conditions (32). For our purpose, we shall inquire more minutely as to the form of these conditions.

11. From (29′) and (27) we have the identity

$$\begin{aligned} &\epsilon_1 r + (\epsilon_1^2 a_2 r^2 + \cdots + \epsilon_1^m a_m r^m) + (\epsilon_1^2 a_{m+2} r^{m+2} + \cdots + \epsilon_1^m a_{2m} r^{2m}) + \cdots \\ &\quad = \epsilon_1[r + (a_2 r^2 + \cdots + a_m r^m) + (a_{m+2} r^{m+2} + \cdots + a_{2m} r^{2m}) + \cdots] \\ &\quad + \lambda_2[r + (a_2 r^2 + \cdots + a_m r^m) + (a_{m+2} r^{m+2} + \cdots + a_{2m} r^{2m}) + \cdots]^2 \\ &\quad + \lambda_3[\qquad\qquad\qquad]^3 + \cdots. \end{aligned} \tag{44}$$

On equating coefficients these yield

$$\begin{aligned} &\lambda_2 = (\epsilon_1^2 - \epsilon_1)a_2, \qquad \lambda_3 = (\epsilon_1^3 - \epsilon_1)a_3 - 2(\epsilon_1^2 - \epsilon_1)a_2^2, \qquad \text{etc.}; \\ &a_2 = \frac{\lambda_2}{\epsilon_1^2 - \epsilon_1}, \qquad a_3 = \frac{\lambda_3}{\epsilon_1^3 - \epsilon_1} + \frac{2\lambda_2^2}{(\epsilon_1^3 - \epsilon_1)(\epsilon_1^2 - \epsilon_1)}, \quad \text{etc.} \end{aligned} \tag{45}$$

Similarly, we obtain corresponding relations for the coefficients μ. Equating the coefficients of z^{pm+1} in (44), and using the set (45), we readily obtain for (32)

$$\begin{aligned} \lambda_{pm+1} &= \frac{2\epsilon_1 - pm}{\epsilon_1^2 - \epsilon_1}\lambda_2\lambda_{pm} + \frac{3\epsilon_1^2 - (pm-1)}{\epsilon_1^3 - \epsilon_1}\lambda_3\lambda_{pm-1} \\ &+ \frac{\begin{array}{r} 2(\epsilon_1^2 + \epsilon_1 - 1)(pm-1) - 2\epsilon_1^2(2\epsilon_1 - 1) \\ - \frac{1}{2}(\epsilon_1 + 1)(pm-1)(pm-2) \end{array}}{(\epsilon_1 + 1)(\epsilon_1^2 - \epsilon_1)^2}\lambda_2^2\lambda_{pm-1} \\ &+ (\text{lower orders of } \lambda), \end{aligned} \tag{46}$$

and similar relations for μ_{sn+1}. The form (46) presupposes $m > 2$.

12. The condition $F = fg$ yields the identity

$$\begin{aligned} K_1 z + K_2 z^2 + K_3 z^3 + \cdots = \quad & \epsilon_2(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots) \\ & + \mu_2(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots)^2 \\ & + \mu_3(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots)^3 \\ & + \cdots\cdots\cdots \end{aligned} \tag{47}$$

If, consistent with conditions (46), and the corresponding ones for the coefficients μ, the λ's and μ's can be chosen so as to satisfy (47) identically, then $F(z)$ can be factored into two periodic transformations of orders m and n respectively. From (47), we have as a necessary condition

$$K_1 = \epsilon_1\epsilon_2. \tag{48}$$

Hence K_1 must be a root of unity. Equating coefficients of higher powers,

$$K_2 = \epsilon_2\lambda_2 + \epsilon_1^2\mu_2, \tag{49}$$

$$K_3 = \epsilon_2\lambda_3 + 2\epsilon_1\lambda_2\mu_2 + \epsilon_1^3\mu_3, \tag{50}$$

$$K_4 = \epsilon_2\lambda_4 + (2\epsilon_1\lambda_3 + \lambda_2^2)\mu_2 + 3\epsilon_1^2\lambda_2\mu_3 + \epsilon_1^4\mu_4, \tag{51}$$

$$K_5 = \epsilon_2\lambda_5 + (2\epsilon_1\lambda_4 + 2\lambda_2\lambda_3)\mu_2 + (3\epsilon_1^2\lambda_3 + 3\epsilon_1\lambda_2^2)\mu_3 + 4\epsilon_1^3\lambda_2\mu_4 + \epsilon_1^5\mu_5, \tag{52}$$

etc. The question now is, assuming (48) to be satisfied, can the λ's and μ's always be chosen without involving necessary conditions among the K's or the indices m and n? Equating the coefficients of z^t, z^{t-1}, z^{t-2}, we obtain

$$\begin{aligned} K_t = & [\epsilon_2\lambda_t + \epsilon_1^t\mu_t] + [2\epsilon_1\mu_2\lambda_{t-1} + \epsilon_1^{t-2}(t-1)\lambda_2\mu_{t-1}] \\ & + [2\lambda_2\mu_2 + 3\epsilon_1^2\mu_3]\lambda_{t-2} \\ & + \left[\epsilon_1^{t-3}(t-2)\lambda_3 + \epsilon_1^{t-4}\frac{(t-2)(t-3)}{2}\lambda_2^2\right]\mu_{t-2} \\ & + [\text{lower orders of } \lambda \text{ and } \mu], \end{aligned} \tag{53}$$

$$\begin{aligned} K_{t-1} = & [\epsilon_2\lambda_{t-1} + \epsilon_1^{t-1}\mu_{t-1}] + [2\epsilon_1\mu_2\lambda_{t-2} + \epsilon_1^{t-3}(t-2)\lambda_2\mu_{t-2}] \\ & + [\text{lower orders of } \lambda \text{ and } \mu], \end{aligned} \tag{54}$$

$$K_{t-2} = [\epsilon_2\lambda_{t-2} + \epsilon_1^{t-2}\mu_{t-2}] + [\text{lower orders of } \lambda \text{ and } \mu]. \tag{55}$$

The entire set of conditions obtained are necessary and sufficient for factorization. We observe that for each coefficient K_t in these equations *two* new quantities λ_t and μ_t are introduced, which may be taken arbitrarily except for periodic orders of these coefficients. Further, the new quantities introduced appear with non-zero coefficients. If either λ_t or μ_t is independent, the condition involving K_t can then be satisfied. Hence, the only way for a necessary condition to obtain among the coefficients K is for *both*

the coefficients λ_t and μ_t to be conditioned in terms of coefficients of lower order in the form (46). Hence t must be of the forms both $pm + 1$ and $sn + 1$. We may write

$$t = pm + 1 = sn + 1. \tag{56}$$

Hence λ_t and μ_t in (53) are not independent. These are then to be replaced by the equivalent expressions in terms of λ's and μ's of lower order, from the form (46). Hence a necessary condition for obtaining a necessary condition among the coefficients K is that it be possible to eliminate simultaneously the coefficients λ_{t-1} and μ_{t-1} from the expressions for K_t and K_{t-1} in terms of these coefficients and those of lower order. Using (56), the condition for this is

$$\frac{\epsilon_2[2\epsilon_1 - (t-1)]}{\epsilon_1^2 - \epsilon_1}\lambda_2 + 2\epsilon_1\mu_2 = \epsilon_2\left\{\frac{\epsilon_1[2\epsilon_2 - (t-1)]}{\epsilon_2^2 - \epsilon_2}\mu_2 + \epsilon_1^{-1}(t-1)\lambda_2\right\},$$

which may be written

$$(t-3)\left[\frac{\lambda_2}{\epsilon_1^2 - \epsilon_1} - \frac{\mu_2}{\epsilon_2^2 - \epsilon_2}\right] = 0. \tag{57}$$

The case $t = 3$ requires $m = n = 2$, already considered by Kasner.* We may then confine our attention to the condition

$$(\epsilon_2^2 - \epsilon_2)\lambda_2 - (\epsilon_1^2 - \epsilon_1)\mu_2 = 0, \tag{58}$$

and choose $t > 3$. If λ_2 and μ_2 can be so chosen as *not* to satisfy (58), then no necessary condition is required among the K's. The only condition upon these is given by (49). Eliminating μ_2, we have

$$K_2 = \frac{\epsilon_1\epsilon_2}{\epsilon_1^2 - \epsilon_1}(\epsilon_1\epsilon_2 - 1)\lambda_2. \tag{59}$$

Here K_2 is assigned and λ_2 completely arbitrary. Hence we may choose λ_2 so that (59) is not satisfied unless

$$K_2 = 0 \qquad \text{and} \qquad \epsilon_1\epsilon_2 - 1 = 0. \tag{60}$$

The latter condition requires $K_1 = 1$. Hence we have the result

If $K_1 \neq 1$, the transformation $F(z) = K_1 z + \cdots$ can always be factored into two transformations of order m and n, the orders being so chosen that $K_1 = \epsilon_1\epsilon_2$, where ϵ_1 and ϵ_2 are primitive roots of unity of order m and n respectively.

13. Suppose now that $K_1 = \epsilon_1\epsilon_2 = 1$. Then we must have

$$m = n, \tag{61}$$

* In this case, $\epsilon_1 = \epsilon_2 = -1$, (45) yield $\lambda_3 = -\lambda_2^2$, $\mu_3 = -\mu_2$, and (49) and (50) give $K_3 - K_2 = 0$ as a necessary condition, (48) requiring $K_1 = 1$. Cf. Kasner, loc. cit.

since ϵ_1 and ϵ_2 are primitive roots. Taking further $K_2 = 0$, we have

$$\mu_2 = -\epsilon_2^3\lambda_2. \tag{62}$$

If we now write (46) and the corresponding expression in μ's in the form

$$\begin{aligned} \lambda_{pm+1} &= \frac{2\epsilon_1 - pm}{\epsilon_1^2 - \epsilon_1}\lambda_2\lambda_{pm} + A\lambda_3\lambda_{pm-1} + B\lambda_2^2\lambda_{pm-1} \\ &\qquad + [\text{lower orders of } \lambda], \\ \mu_{sn+1} &= \frac{2\epsilon_2 - sn}{\epsilon_2^2 - \epsilon_2}\mu_2\mu_{sn} + C\mu_3\mu_{sn-1} + D\mu_2^2\mu_{sn-1} \\ &\qquad + [\text{lower orders of } \mu], \end{aligned} \tag{63}$$

and eliminate λ_{t-1} and μ_{t-1} from equations (53) and (54), we have, on using (56), (60), (61) and (62),

$$\begin{aligned} K_t &+ \frac{t-3}{\epsilon_1^2 - \epsilon_1}\lambda_2 K_{t-1} \\ &= \left[\epsilon_2 A\lambda_3 + \epsilon_2 B\lambda_2^2 - 2\epsilon_2^5\lambda_2^2 + 3\epsilon_1^2\mu_3 - \frac{2\epsilon_2^2(t-3)}{\epsilon_1^2 - \epsilon_1}\lambda_2^2\right]\lambda_{t-2} \\ &\quad + \Bigg[\epsilon_1 C\mu_3 + \epsilon_2^5 D\lambda_2^2 + \epsilon_2^2(t-2)\lambda_3 \\ &\qquad\qquad + \epsilon_2^3\frac{(t-2)(t-3)}{2}\lambda_2^2 + \frac{\epsilon_1^2(t-2)(t-3)}{\epsilon_1^2 - \epsilon_1}\Bigg]\mu_{t-2} \\ &\quad + [\text{lower orders of } \lambda \text{ and } \mu]. \end{aligned} \tag{64}$$

Equation (55) becomes

$$K_{t-2} = [\epsilon_2\lambda_{t-2} + \epsilon_2\mu_{t-2}] + [\text{lower orders of } \lambda \text{ and } \mu]. \tag{65}$$

The form of (64) presupposes $t > 5$. For the moment we shall assume this satisfied. Here λ_{t-2} and μ_{t-2} in (64) and (65) can be so chosen as to satisfy both, unless the determinant of these coefficients vanishes. The condition for this becomes

$$[\epsilon_2 A - \epsilon_2^2(t-2)]\lambda_3 + [3\epsilon_1^2 - \epsilon_1 C]\mu_3 + \frac{2\epsilon_2(t-5)}{\epsilon_1^2 - 1}\lambda_2^2 = 0.$$

Here λ_2 is independent and the only condition upon λ_3 and μ_3 is (50). Eliminating μ_3 we obtain

$$L\lambda_3 + M\lambda_2^2 + NK_3 = 0, \tag{66}$$

where L, M and N are rational expressions in t and ϵ_2. We have

$$N = \epsilon_2\frac{t-5}{\epsilon_2^2 - 1} \neq 0.$$

In (66), λ_2 and λ_3 are completely independent. Hence these may be chosen

so as not to satisfy (66), unless

$$L = M = 0 \tag{67}$$

identically, and $K_3 = 0$. On substituting their values for A and C, we find these conditions fulfilled. Hence if $K_1 = 1$, and $K_2 = K_3 = 0$, we must resort to further analysis.

14. For this purpose we return to the identity (44). If t be *even*, we may write the general condition in the form

$$\begin{aligned} \epsilon_1^t a_t = {} & \epsilon_1 a_t + \lambda_t \\ & + \lambda_2 M_{t-1} + \lambda_{t-1} M_2 \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{k+1} M_{t-k} + \lambda_{t-k} M_{k+1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{t/2} M_{(t/2)+1} + \lambda_{(t/2)+1} M_{t/2}, \end{aligned} \tag{68}$$

where

$$M_{t-k} = (k+1) a_{t-k} + (\text{polynomial in lower orders of } a). \tag{69}$$

If t be *odd*, we have the form

$$\begin{aligned} \epsilon_1^t a_t = {} & \epsilon_1 a_t + \lambda_t \\ & + \lambda_2 M_{t-1} + \lambda_{t-1} M_2 \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{k+1} M_{t-k} - \lambda_{t-k} M_{k+1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{(t-1)/2} M_{(t+3)/2} + \lambda_{(t+3)/2} M_{(t-1)/2} \\ & + \lambda_{(t+1)/2} M_{(t+1)/2}. \end{aligned} \tag{70}$$

From (45), we have a_2 and a_3 expressed in terms of the corresponding λ's. Substituting in (69) for $t = 4$, we have a_4 in terms of λ's. Using (68) and (70), we finally have all a's that appear in (44) expressed in the forms (68) and (70) where the M's are expressed in λ's in the form

$$\begin{aligned} M_{t-k} &= \frac{k+1}{\epsilon_1^{t-k} - \epsilon_1} \lambda_{t-k} + (\text{polynomial in lower orders of } \lambda), \\ M_{k+1} &= \frac{t-k}{\epsilon_1^{k+1} - \epsilon_1} \lambda_{k+1} + (\text{polynomial in lower orders of } \lambda). \end{aligned} \tag{71}$$

If $t - k \equiv 1 \pmod m$, the first term indicated in (71) is of course absent. The highest order of λ that multiplies λ_{t-k} is λ_{k+1}, and occurs only in the products $\lambda_{k+1} M_{t-k}$ and $\lambda_{t-k} M_{k+1}$. If $t - k > (t+1)/2$, then λ_{t-k} occurs only to the first power in (68) or (70). The coefficient of λ_{t-k} is

$$\left(\frac{k+1}{\epsilon_1^{t-k} - \epsilon_1} + \frac{t-k}{\epsilon_1^{k+1} - \epsilon_1} \right) \lambda_{k+1} + (\text{polynomial in lower orders of } \lambda).$$

If now we take $t-1$ a multiple of m, we obtain from (68), if t be even,

$$\lambda_t = N_2\lambda_{t-1} + N_3\lambda_{t-2} + \cdots + N_{k-1}\lambda_{t-k} + \cdots + N_{t/2}\lambda_{(t/2)+1} \\ + \text{(polynomial in lower orders of } \lambda), \tag{72}$$

and from (70), if t be odd,

$$\lambda_t = N_2\lambda_{t-1} + N_3\lambda_{t-2} + \cdots + N_{k+1}\lambda_{t-k} + \cdots + \tfrac{1}{2}N_{(t+1)/2}\lambda_{(t+1)/2} \\ + \text{(polynomial in lower orders of } \lambda), \tag{73}$$

where

$$N_{k+1} = \frac{\epsilon_1^k(k+1) - (t-k)}{\epsilon_1^{k+1} - \epsilon_1}\lambda_{k+1} + \text{(polynomial in lower orders of } \lambda). \tag{74}$$

If $k \equiv 0 \pmod{m}$, then the term of order $k+1$ in (74) is missing. If $t = m+1$, every term is present. The relations (72) and (73) are simply more explicit forms of the necessary conditions (32). In similar fashion, we may now write out the corresponding expressions in the μ's, thus obtaining the necessary relations among the coefficients of f and g in (43).

15. We shall now write (53) in a form corresponding to (72) and (73), and then apply once more the method of Sections 12 and 13. Referring to the identity (47) we may write, if t is *even*,

$$\begin{aligned} K_t = {} & \epsilon_2\lambda_t + \epsilon_1^t\mu_t \\ & + P_{t,\,2}\lambda_{t-1} + Q_{t,\,2}\mu_{t-1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + P_{t,\,k-1}\lambda_{t-k} + Q_{t,\,k+1}\mu_{t-k} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + P_{t,\,t/2}\lambda_{(t/2)+1} + Q_{t,\,t/2}\mu_{(t/2)+1} \\ & + \text{(polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t/2)+1), \end{aligned} \tag{75}$$

and if t is *odd*,

$$\begin{aligned} K_t = {} & \epsilon_2\lambda_t + \epsilon_1^t\mu_t \\ & + P_{t,\,2}\lambda_{t-1} + Q_{t,\,2}\mu_{t-1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + P_{t,\,k+1}\lambda_{t-k} + Q_{t,\,k+1}\mu_{t-k} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + P_{t,\,(t-1)/2}\lambda_{(t+3)/2} + Q_{t,\,(t-1)/2}\mu_{(t+3)/2} + Q_{t,\,(t+1)/2}\mu_{(t+1)/2} \\ & + \text{(polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2), \end{aligned} \tag{76}$$

where

$$P_{t,\,k+1} = (k+1)\epsilon_1^k\mu_{k+1} + \text{(lower orders of } \lambda \text{ and } \mu) \tag{77}$$

and

$$Q_{t,\,k+1} = (t-k)\epsilon_1^{t-k-1}\lambda_{k+1} + \text{(lower orders of } \lambda). \tag{78}$$

All λ's and μ's in (75) and (76) are independent except those of order $pm+1$ or $sn+1$. We now examine the cases determined by (56), (60),

(61), and (62). In that event λ_t and μ_t in (75) and (76) must be replaced by the equivalent expressions of the form (72) or (73). If $t = n + 1$, then all the remaining λ's and μ's in (75) or (76) are independent. The coefficient of λ_{k+1} in (78) becomes $(t-k)\epsilon_1^{-k}$, or $(t-k)\epsilon_2^k$.

Substituting for λ_t and μ_t in (75) their equivalent expressions, we have for t even

$$\begin{aligned} K_t = E_2\lambda_{t-1} &+ F_2\mu_{t-1} \\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &+ E_{k+1}\lambda_{t-k} + F_{k+1}\mu_{t-k} \\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &+ E_{t/2}\lambda_{(t/2)+1} + F_{t/2}\mu_{(t/2)+1} \\ &+ (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t/2)+1), \end{aligned} \tag{79}$$

and from (76) for t odd

$$\begin{aligned} K_t = E_2\lambda_{t-1} &+ F_2\mu_{t-1} \\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &+ E_{k+1}\lambda_{t-k} + F_{k+1}\mu_{t-k} \\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &+ E_{(t-1)/2}\lambda_{(t+3)/2} + F_{(t-1)/2}\mu_{(t+3)/2} \\ &+ \left(\frac{\epsilon_2}{2}N_{(t+1)/2}\lambda_{(t+1)/2} + Q_{t,\,(t+1)/2}\mu_{(t+1)/2} + \frac{\epsilon_1}{2}T_{(t+1)/2}\mu_{(t+1)/2}\right) \\ &+ (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2), \end{aligned} \tag{80}$$

where

$$E_{k+1} = P_{t,\,k+1} + \epsilon_2 N_{k+1}; \quad F_{k+1} = Q_{t,\,k+1} + \epsilon_1 T_{k+1}, \tag{81}$$

where T is the expression in μ's corresponding to N in (74), and has the form

$$T_{k+1} = \frac{\epsilon_2^k(k+1) - (t-k)}{\epsilon_2^{k+1} - \epsilon_2}\mu_{k+1} + (\text{polynomial in lower orders of } \mu). \tag{82}$$

16. From the form of (75) and (76) we may write

$$\begin{aligned} K_{t-1} = \epsilon_2\lambda_{t-1} &+ \mu_{t-1} \\ &+ P_{t-1,\,2}\lambda_{t-2} + Q_{t-1,\,2}\mu_{t-2} \\ &+ P_{t-1,\,3}\lambda_{t-3} + Q_{t-1,\,3}\mu_{t-3} \\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \end{aligned} \tag{83}$$

and similar expressions for the lower orders of K.

Comparing (83) with (79) or (80), we observe that K_t may be taken independent of the lower orders of K unless

$$\begin{vmatrix} E_2 & F_2 \\ \epsilon_2 & 1 \end{vmatrix} = 0 \tag{84}$$

identically. As already seen, this reduces to (57), involving (60), (61) and (62). Assuming (84) satisfied, and eliminating λ_{t-1} and μ_{t-1} from the expressions for K_t and K_{t-1}, the eliminant and the expression for K_{t-2} can always be satisfied by a proper choice of the λ's and μ's involved unless λ_{t-2} and μ_{t-2} can be eliminated simultaneously. This obviously requires that the coefficients of λ_{t-1}, μ_{t-1}, λ_{t-2}, and μ_{t-2} in the expressions for K_t, K_{t-1} and K_{t-2} shall be linearly dependent. The condition for this is that all 3-rowed determinants of the matrix

$$\left\| \begin{matrix} E_2 & F_2 & E_3 & F_3 \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} \end{matrix} \right\| \tag{85}$$

vanish.

Assuming (84) satisfied, the second column of (85) may be replaced by 0's, leaving but one determinant for consideration. Since E_3 and F_3 are the only expressions containing λ's and μ's of order 3, we have as a necessary condition, on expanding according to the minors of the elements of the first column and using (62),

$$\epsilon_1^{-1}E_3 - \epsilon_2 F_3 + (\text{polynomial in } \lambda_2) = 0,$$

the λ-polynomial being in fact λ_2^2 multiplied by a constant. Using (74), (82), (77), (78) and (81), this reduces to

$$\frac{t-5}{\epsilon_1^2 - 1}(\epsilon_2\lambda_3 + \epsilon_1^3\mu_3) + (\text{polynomial in } \lambda_2) = 0. \tag{86}$$

Here λ_2 is completely independent and λ_3 and μ_3 restricted only by the relation (50). Assuming for the moment that $t \neq 5$, and using (50), the last equation becomes

$$K_3 + (\text{polynomial in } \lambda_2) = 0. \tag{87}$$

Here K_3 is assigned. Hence λ_2 may be so chosen that (88) is *not* satisfied, unless $K_3 = 0$ and the polynomial in λ_2 in (87) vanishes identically. Both of these conditions are necessary if there is to be a necessary relation among the K's. In Section 13 we found that the polynomial of (87) does vanish identically.

17. Assuming now $K_2 = K_3 = 0$, and following the same reasoning, λ_{t-1}, λ_{t-2}, λ_{t-3} and the corresponding μ's may be chosen so that the conditions for K_t, K_{t-1}, K_{t-2} and K_{t-3} are all satisfied unless the coefficients of the λ's and μ's in these expressions are linearly dependent. As before, the condition for this is that the matrix

$$\left\| \begin{matrix} E_2 & F_2 & E_3 & F_3 & E_4 & F_4 \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} & P_{t-1,\,3} & Q_{t-1,\,3} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} & P_{t-2,\,2} & Q_{t-2,\,2} \\ 0 & 0 & 0 & 0 & \epsilon_2 & \epsilon_1^{-2} \end{matrix} \right\| \tag{88}$$

be of rank < 4. Since in (84) and (85) the last columns are linearly expressible in terms of the previous columns, we may replace columns 2 and 4 in (88) by 0's, leaving but one 4-rowed determinant for consideration. In (88), E_4 and F_4 are the only elements involving λ's and μ's of order 4. Expanding in terms of the determinants of the last two columns, and replacing μ_2 and μ_3 by their equivalent expressions in λ by putting $K_2 = K_3 = 0$ in (49) and (50),

$$\epsilon_2^2 \begin{vmatrix} E_4 & F_4 \\ \epsilon_2 & \epsilon_1^{-2} \end{vmatrix} + (\text{polynomial in } \lambda\text{'s of order} < 4) = 0.$$

This reduces to

$$\frac{t-7}{\epsilon_1^3 - 1}(\epsilon_2\lambda_4 + \epsilon_1^4\mu_4) + (\text{polynomial in } \lambda\text{'s of order} < 4) = 0. \qquad (89)$$

If $t \neq 7$, this becomes, on using (51),

$$K_4 + (\text{polynomial in } \lambda\text{'s of order} < 4) = 0. \qquad (90)$$

As before, the λ's can be chosen so that (90) is not satisfied, unless $K_4 = 0$ and the λ-polynomial vanishes identically. If $K_4 = 0$, (51) gives μ_4 also in terms of λ's.

Suppose now that the condition that K_t depend for its value on K's of lower order requires

$$K_2 = K_3 = K_4 = \cdots = K_k = 0. \qquad (91)$$

These determine $\mu_2, \mu_3, \cdots, \mu_k$ as polynomials in λ's of corresponding and lower orders. Continuing as before, we find as another necessary condition that the matrix

$$\left\| \begin{matrix} E_2 & F_2 & E_3 & F_3 & \cdots & E_{k+1} & F_{k+1} \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} & \cdots & P_{t-1,\,k} & Q_{t-1,\,k} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} & \cdots & P_{t-2,\,k-1} & Q_{t-2,\,k-1} \\ 0 & 0 & 0 & 0 & \cdots & P_{t-3,\,k-2} & Q_{t-3,\,k-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \epsilon_2 & \epsilon_1^{1-k} \end{matrix} \right\| \qquad (92)$$

be of rank $< k + 1$. As before, every second column except the last may be replaced by 0's, leaving but a single $(k + 1)$-rowed determinant for consideration. Expanding according to the elements of the last two columns we obtain

$$\epsilon_2^{k-1} \begin{vmatrix} E_{k+1} & F_{k+1} \\ \epsilon_2 & \epsilon_1 \end{vmatrix} + (\text{polynomial in } \lambda\text{'s of order} < k + 1) = 0,$$

which reduces to

$$\frac{t-2k-1}{\epsilon_1^k - 1}(\epsilon_2\lambda_{k+1} + \epsilon_1^{k+1}\mu_{k+1}) + (\text{polynomial in } \lambda\text{'s of order} < k + 1) = 0,$$

or

$$\frac{t-2k-1}{\epsilon_1^k-1}K_{k+1}+(\text{polynomial in } \lambda\text{'s of order} < k+1)=0. \tag{93}$$

18. We shall now first restrict ourselves to the case $t = n + 1$. Hence all the terms indicated in (74) and (82) are present, and all λ's and μ's in (75) or (76), except λ_t and μ_t, are independent. We shall find it convenient to consider two cases, according as t is *even* or *odd*.

We shall first consider t *even*. Then $t \neq 2k + 1$ in any case. From the form of (79), we may continue the method of the previous sections up to the case $k = t/2$. Hence, if the condition $K_2 = K_3 = \cdots = K_k = 0$ has been shown necessary, we obtain from (93) the condition $K_{k+1} = 0$ and the further condition that the polynomial in λ vanish identically. Hence we have as a necessary condition that K_t be conditioned in terms of lower orders of K

$$K_2 = K_3 = \cdots = K_{t/2} = 0. \tag{94}$$

If the polynomials indicated in (87) and (90) and that resulting from (93) do not vanish, then of course all K's are independent. However, we shall presently show by an indirect method that they all vanish identically. From (94) and the relations of Section 12, we have all μ's up to order $t/2$ determined as polynomials in λ's of corresponding and lower orders.

Conditions (94) and the vanishing of the polynomials considered above suffice for the elimination of the λ's and μ's of orders $(t/2) + 1$ to $t - 1$ from the expressions for $K_t, K_{t-1}, \cdots, K_{(t/2)+1}$. From the method of procedure in Section 13 we observe that, if we eliminate λ_{t-1} and μ_{t-1} from between K_t and K_{t-1} and then λ_{t-2} and μ_{t-2} from between the result and K_{t-2}, etc., the result of the entire elimination of λ's and μ's of orders $t - 1$ down to order $(t/2) + 1$ takes the form

$$K_t + H_2K_{t-1} + H_3K_{t-2} + \cdots + H_{t/2}K_{(t/2)+1} = P(\lambda), \tag{95}$$

where $P(\lambda)$ is a polynomial in λ's of order $< (t/2) + 1$. It is also clear that no λ of order $> t/2$ appears in the left member of (95). From the form of (79), (83), etc., we observe that the highest orders of λ and μ in the coefficients E and F following λ_{t-1} and μ_{t-1} are not affected by the elimination of these variables; that those following λ_{t-2} and μ_{t-2} are not affected by the next elimination, etc. In other words, the term of highest order in H_{k+1} in (95) is ϵ_1 times the corresponding term in the coefficient of λ_{t-k} in (79). In other words,

$$H_{k+1}=\frac{t-2k-1}{\epsilon_1^{k+1}-\epsilon_1}\lambda_{k+1}+(\text{polynomial in } \lambda\text{'s of lower order}). \tag{96}$$

The coefficient of λ_{k+1} in (96) is, furthermore, not zero. In the left member of (95), $\lambda_{t/2}$ occurs only in $H_{t/2}$. If it occurs in $P(\lambda)$, it is multiplied by some variable λ. Hence if $K_{(t/2)+1} \neq 0$, the λ's of order $< t/2$ may be assigned arbitrarily and $\lambda_{t/2}$ assigned a value to satisfy (95) for *all* values of the K's. Hence for a *necessary* relation to obtain among the K's, we must have $K_{(t/2)+1} = 0$. Similarly, we must have $K_{(t/2)+2} = 0$, etc. Hence the only way in which a necessary condition must subsist among the K's is to have

$$K_{(t/2)+1} = K_{(t/2)+2} = \cdots = K_{t-1} = 0.$$

If, now, $P(\lambda)$ in (95) vanish identically, the necessary resulting condition would be

$$K_t = 0. \tag{97}$$

Hence, if t is even and $= n + 1$, the only condition that may be required among the K's is to have $K_t = 0$, and this can result only if

$$K_2 = K_3 = \cdots = K_{t-1} = 0,$$

and $P(\lambda)$ vanish identically.

These are, furthermore, sufficient conditions if the polynomials resulting from (93) also vanish identically. If the procedure be followed out in detail for $t = 4$, we obtain (95) in the form

$$K_4 + \frac{\epsilon_1^2\lambda_2}{\epsilon_1 - 1} K_3 = 0. \tag{98}$$

For the case $t = 6$ we obtain, without any essential difficulty,

$$K_6 + \frac{3\lambda_2}{\epsilon_1^2 - \epsilon_1} K_5 + \left[\frac{\epsilon_1^4}{\epsilon_1^2 - 1}\lambda_3 + \frac{2\epsilon_1^3(\epsilon_1 + 2)}{(\epsilon_1 - 1)(\epsilon_1^2 - 1)}\lambda_2^2\right] K_4 = 0. \tag{99}$$

19. Suppose now that t is *odd*. The method of Section 17 can be applied so long as t remains $> 2k + 1$. Hence we obtain as necessary conditions for dependence among the K's

$$K_2 = K_3 = \cdots = K_{(t-1)/2} = 0.$$

Also, from the preceding discussion, the elimination of K's of all orders down to $K_{(t+3)/2}$ does not affect the leading terms of order $(t + 1)/2$ in (80). Evaluating the corresponding expressions in (80),

$$\begin{aligned}\frac{\epsilon_2}{2} N_{(t+1)/2}\lambda_{(t+1)/2} &+ Q_{t,\,(t+1)/2}\mu_{(t+1)/2} + \frac{\epsilon_1}{2} T_{(t+1)/2}\mu_{(t+1)/2} \\ &= \frac{t+1}{4}(\epsilon_2\lambda_{(t+1)/2} - \epsilon_1\mu_{(t+1)/2})^2 + \lambda_{(t-1)/2}\text{ (lower orders)} \\ &\qquad + \mu_{(t+1)/2}\text{ (lower orders)}.\end{aligned} \tag{100}$$

From the form of (75) and (76),

$$K_{(t+1)/2} = \epsilon_2\lambda_{(t+1)/2} - \epsilon_1\mu_{(t+1)/2} + \text{(polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2). \tag{101}$$

Hence the result of the elimination of K's down to order $(t+3)/2$ takes the form

$$\begin{aligned} K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2} \\ = \lambda_{(t-1)/2}\text{ (lower orders)} + \mu_{(t+1)/2}\text{ (lower orders)} \\ + \text{(polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2). \end{aligned} \tag{102}$$

Here $\lambda_{(t+1)/2}$ and $\mu_{(t+1)/2}$ are not independent, being connected by (101). The question now arises, can these variables be eliminated from between (102) and (101)? For this purpose we have to evaluate the coefficients of $\lambda_{(t+1)/2}$ and $\mu_{(t+1)/2}$ in (102).

20. We shall, for convenience, introduce

$$e = \frac{t+1}{2}, \tag{103}$$

whence

$$t = 2e - 1; \qquad \epsilon_1^{e-1} = \epsilon_2^{e-1} = -1. \tag{104}$$

We shall now trace back all products of the form $\lambda_e\lambda_{e-1}$, $\lambda_e\mu_{e-1}$, $\mu_e\lambda_{e-1}$, $\mu_e\mu_{e-1}$, occurring in the process of obtaining (102). In the following, all expressions involving such products will be included.

From (44), we have the general form

$$\begin{aligned} \epsilon^t a_t = \epsilon a_t &+ \lambda_t \\ &+ \lambda_2[2a_{t-1} + \cdots + 2a_{e-1}a_{e+1}] \\ &+ \cdots\cdots\cdots\cdots \\ &+ \lambda_{e-1}[(e-1)a_{e-1} + (e-1)(e-2)a_2a_e + \text{(lower orders)}] \\ &+ \lambda_e[ea_e + e(e-1)a_2a_{e-1} + \text{(lower orders)}] \\ &+ \lambda_{e+1}[(e+1)a_{e-1} + \text{(lower orders)}] \\ &+ \cdots\cdots\cdots\cdots \\ &+ \lambda_{t-1}[(t-1)a_2]. \end{aligned} \tag{105}$$

From the general form (105) we also have

$$\begin{aligned} (\epsilon^{t-1} - \epsilon)a_{t-1} = \lambda_{t-1} + \cdots + \lambda_{e-1}[(e-1)a_e + \text{(lower orders)}] \\ + \lambda_e[ea_{e-1} + \text{(lower orders)}] + \cdots; \end{aligned} \tag{106}$$

$$(\epsilon^{e+1} - \epsilon)a_{e+1} = \lambda_{e+1} + \lambda_2[2a_e + \text{(lower orders)}] + \cdots + \lambda_e e a_2; \tag{107}$$

$$\begin{aligned} (\epsilon^e - \epsilon)a_e = \lambda_e + \lambda_2[2a_{e-1} + \text{(lower orders)}] + \cdots \\ + \lambda_{e-1}(e-1)a_2; \end{aligned} \tag{108}$$

$$(\epsilon^{e-1} - \epsilon)a_{e-1} = \lambda_{e-1} + \text{(lower orders)}. \tag{109}$$

Formulas (105) and (107) presuppose $t > 5$. The case $t = 5$ will be considered specially.

Using (106), (107), (108), (109) and (45), we have from (105)

$$\begin{aligned}\lambda_t = \lambda_2\lambda_{t-1}&\left[\frac{2}{\epsilon_1 - 1} - \frac{t-1}{\epsilon_1(\epsilon_1 - 1)}\right] \\ &+ \cdots + \lambda_{e-1}\lambda_{e+1}\left[\frac{e}{\epsilon_1} + \frac{\epsilon_1 - 1}{\epsilon_1(\epsilon_1 + 1)}\right] \\ &+ \frac{e}{2\epsilon_1}\lambda_e^2 + \lambda_2\lambda_e\lambda_{e-1}\left[\frac{2e(e-1)}{\epsilon_1^2(\epsilon_1 - 1)} - \frac{2e(2\epsilon_1 + 1) - 4}{\epsilon_1(\epsilon_1^2 - 1)}\right] + \cdots.\end{aligned} \tag{110}$$

We may now proceed to the corresponding expression for μ_t. Since K_{e-1} is presumed zero,

$$0 = \epsilon_2\lambda_{e-1} - \mu_{e-1} + \text{(lower orders)},$$

whence

$$\mu_{e-1} = \epsilon_2\lambda_{e-1} + \text{(lower orders of } \lambda). \tag{111}$$

We may also write

$$K_{e+1} = [\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}] - \lambda_2[2\epsilon_2^2\lambda_e + e\mu_e] + \text{(lower orders } \lambda), \tag{112}$$

$$K_e = [\epsilon_2\lambda_e - \epsilon_1\mu_e] - (e+1)\epsilon_2^2\lambda_2\lambda_{e-1} + \text{(lower orders } \lambda), \tag{113}$$

$$K_e^2 = [\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 - 2\epsilon_2^2(e+1)\lambda_2\lambda_{e-1}[\epsilon_2\lambda_e - \epsilon_1\mu_e] + \text{(lower orders } \lambda). \tag{114}$$

The expression for μ_t becomes, since $\epsilon_1\epsilon_2 = 1$,

$$\begin{aligned}\mu_t = \lambda_2\mu_{t-1}&\left[\frac{2\epsilon_2^2}{\epsilon_1 - 1} - \frac{\epsilon_2(t-1)}{\epsilon_1 - 1}\right] + \cdots + \lambda_{e-1}\lambda_{e+1}\left[e - \frac{\epsilon_1 - 1}{\epsilon_1 + 1}\right] \\ &+ \frac{\epsilon_1 e}{2}\mu_e^2 + \epsilon_2\lambda_2\lambda_{e-1}\mu_e\left[\frac{2e(e-1)}{\epsilon_1 - 1} - \frac{2e(2\epsilon_2 + 1) - 4}{\epsilon_1^2 - 1}\right] + \cdots.\end{aligned} \tag{115}$$

21. Referring now to (47), we may write

$$\begin{aligned}K_t = [\epsilon_2\lambda_t + \epsilon_1\mu_t] &+ [-2\epsilon_2^2\lambda_2\lambda_{t-1} + \epsilon_2(t-1)\lambda_2\mu_{t-1}] + \cdots \\ &- \epsilon_2^3\lambda_2[2\lambda_{e-1}\lambda_e + \cdots] + \cdots \\ &+ \mu_{e+1}[(e+1)\epsilon_1^e\lambda_{e-1} + \cdots] \\ &+ \mu_e[e\epsilon_1^{e-1}\lambda_e + e(e-1)\epsilon_1^{e-2}\lambda_2\lambda_{e-1} + \cdots] \\ &+ \mu_{e-1}[(e-1)\epsilon_1^{e-2}\lambda_{e+1} + (e-1)(e-2)\epsilon_1^{e-3}\lambda_2\lambda_e + \cdots] + \cdots,\end{aligned}$$

which may be written

$$\begin{aligned}K_t = [\epsilon_2\lambda_t + \epsilon_1\mu_t] &+ \lambda_2[-2\epsilon_2^2\lambda_{t-1} + \epsilon_2(t-1)\mu_{t-1}] + \cdots \\ &+ \lambda_{e-1}[-\epsilon_2^2(e-1)\lambda_{e+1} - \epsilon_1(e+1)\mu_{e+1}] - e\lambda_e\mu_e \\ &+ \lambda_2\lambda_{e-1}\{\lambda_e[-2\epsilon_2^3 - \epsilon_2^3(e-1)(e-2)] \\ &\qquad + \mu_e[-\epsilon_2 e(e-1)]\} + \cdots.\end{aligned} \tag{116}$$

The only other K involving like products is

$$K_{t-1} = [\epsilon_2\lambda_{t-1} + \mu_{t-1}] + \cdots + \lambda_{e-1}[-\epsilon_2^2(e-1)\lambda_e - e\lambda_e] + \cdots. \tag{117}$$

Substituting now from (110) and (115) in (116),

$$K_t = -\epsilon_2 \frac{t-3}{\epsilon_1 - 1}\lambda_2[\epsilon_2\lambda_{t-1} + \mu_{t-1}] + \cdots$$
$$+ \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + \frac{2\lambda_{e-1}}{\epsilon_1 + 1}[\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}]$$
$$+ \epsilon_2^2\lambda_2\lambda_{e-1}\left\{\begin{array}{l} \epsilon_2\lambda_e\left[-\dfrac{\epsilon_1 - 3}{\epsilon_1 - 1}e^2 - \dfrac{\epsilon_1^2 + 4\epsilon_1 + 5}{\epsilon_1^2 - 1}e \right. \\ \qquad \left. + \dfrac{4}{\epsilon_1^2 - 1}(-\epsilon_1^2 + \epsilon_1 + 1)\right] \\ + \epsilon_1\mu_e\left[\dfrac{\epsilon_1 + 1}{\epsilon_1 - 1}e^2 - \dfrac{\epsilon_1^2 + 4\epsilon_1 + 5}{\epsilon_1^2 - 1}e \right. \\ \qquad \left. + \dfrac{4\epsilon_1}{\epsilon_1^2 - 1}\right] \end{array}\right\} + \cdots. \tag{118}$$

From (118) and (117), we now have

$$K_t + \frac{\epsilon_2(t-3)}{\epsilon_1 - 1}\lambda_2 K_{t-1} = [\text{orders} < t - 1 \text{ and } > e + 1]$$
$$+ \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + \frac{2\lambda_{e-1}}{\epsilon_1 + 1}[\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}]$$
$$+ \epsilon_2^2\lambda_2\lambda_{e-1}\left\{\begin{array}{l} \epsilon_2\lambda_e\left[-e^2 - \dfrac{\epsilon_1^2 - 2\epsilon_1 - 1}{\epsilon_1^2 - 1}e \right. \\ \qquad \left. - \dfrac{4\epsilon_1^2}{\epsilon_1^2 - 1}\right] \\ + \epsilon_1\mu_e\left[e^2 - \dfrac{\epsilon_1^2 + 1}{\epsilon_1^2 - 1}e + \dfrac{4\epsilon_1}{\epsilon_1^2 - 1}\right] \end{array}\right\} + \cdots. \tag{119}$$

We may observe that the right member of (119) is not affected by the elimination of any more λ's and μ's down to those of order $e + 1$. From (96),

$$H_{(t-1)/2} = -\frac{2\lambda_{e-1}}{\epsilon_1 + 1} + (\text{lower orders of } \lambda). \tag{120}$$

In the process of obtaining (102) we have then, from the form (119) and (112),

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2}$$
$$= \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + [\text{polynomial in } \lambda\text{'s of order} < e] \tag{121}$$
$$- \frac{\epsilon_2^2\lambda_2\lambda_{e-1}}{\epsilon_1^2 - 1}[e^2(\epsilon_1^2 - 1) + (\epsilon_1^2 - 2\epsilon_1 - 1)e + 4\epsilon_1][\epsilon_2\lambda_e - \epsilon_1\mu_e].$$

From (121) and (114),

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2}$$
$$= \epsilon_2 \frac{t-3}{\epsilon_1^2 - 1}\lambda_2\lambda_{e-1}[\epsilon_2\lambda_e - \epsilon_1\mu_e] + (\text{polynomial in } \lambda\text{'s of order} < \epsilon). \tag{122}$$

Using (113), (122) may now be written

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2}$$
$$- \epsilon_2 \frac{t-3}{\epsilon_1^2 - 1}\lambda_2\lambda_{(t-1)/2}K_{(t+1)/2} = P(\lambda), \tag{123}$$

where $P(\lambda)$ is some polynomial in λ's of order $< (t+1)/2$. Equation (123) corresponds to (102). For the special case $t = 5$, we obtain readily

$$K_5 + (\epsilon_1 - 1)\lambda_2K_4 - \frac{3}{2}K_3^2 - \frac{\epsilon_1}{2}\lambda_2^2K_3 = 0. \tag{124}$$

In (123), $\lambda_{(t-1)/2}$ occurs in the left member only in the coefficients of $K_{(t+3)/2}$ and $K_{(t+1)/2}$. In the latter it occurs in product with the independent λ_2. Hence, reasoning as before, (123) can be made to hold for all K's, unless $K_{(t+1)/2} = 0$ and $K_{(t+3)/2} = 0$. Similarly, we must require

$$K_{(t+5)/2} = \cdots = K_{t-1} = 0.$$

Only in that event, and if $P(\lambda) = 0$ identically, can we obtain the necessary condition $K_t = 0$.

We have seen that if $K_1 \neq 1$ in (43), then F is always factorable as the product of two periodic transformations. If $K_1 = 1$ and F is factorable, the periods of f and g in (43) must be equal, giving $m = n$. We may now state the result

If the transformation

$$F(z) = z + K_2z^2 + K_3z^3 + \cdots$$

is to be factored into two transformations of period n, then the only condition that may be found necessary among the K's is of the form

$$K_{sn+1} = \textit{linear expression in } K\textit{'s of lower order;}$$

if s is to $= 1$, the only way in which the condition can arise is to have

$$K_2 = K_3 = \cdots = K_n = 0,$$

in which event the condition is

$$K_{n+1} = 0.$$

In other words, if $F(z)$ takes the form

$$z + K_r z^r + \cdots,$$

then F can always be factored into two transformations of period $> r - 1$.

It should be remembered that we have not as yet shown that if the condition

$$K_2 = K_3 = \cdots = K_n = 0 \tag{125}$$

holds, then $K_{n+1} = 0$ necessarily. In other words, we have to show that $P(\lambda)$ in (95) and (123) and the λ-polynomials resulting from (93) all vanish identically. We have besides to consider the more general case where $t = sn + 1$, $s > 1$.

22. We shall consider the former question first. It should be observed that $P(\lambda)$ in (95) and (123), and the other λ-polynomials we have presumed to vanish are independent of K_{n+1} or any K of higher order. Hence, if it can be shown that the function

$$y = z + z^{n+1}; \qquad n > 2 \tag{126}$$

cannot be factored into two transformations of period n, then it would follow that $P(\lambda) = 0$ identically, and that the other λ-polynomials vanish correspondingly.

23. We have seen, Theorem V, that if in (36)

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, 3, \cdots,$$

then there is one and only one transformation (36) which will put (34) into the form (35). Suppose now that F in (43) takes the form (126) and we require f and g as defined in (43) to satisfy

$$g[f(z)] = z + z^{n+1}. \tag{127}$$

As we have seen, we must have $\epsilon_1\epsilon_2 = 1$, and $m = n$. Furthermore, from (43) and (127) we observe, on writing out the detailed conditions on λ's and μ's, that each coefficient μ is determined uniquely as a polynomial in λ's of corresponding and lower orders. Hence, *for all orders* $< n + 1$, the coefficients μ have the same relations to the coefficients λ as would be obtained from (43) and the condition

$$g[f(z)] = z. \tag{128}$$

This condition would require $g = f^{-1}$. Furthermore, there exists a unique function

$$h(z) = z + \sum_{s=0}^{\infty} \sum_{r=1}^{n-1} A_{sn+r+1} z^{sn+r+1} \tag{129}$$

such that, symbolically,

$$f = h^{-1}\epsilon_1 h. \tag{130}$$

From (130), $f^{-1} = h^{-1}\epsilon_1^{-1}h$. Hence $g(z)$ as determined by (128) may be written

$$g = h^{-1}\epsilon_1^{-1}h. \tag{131}$$

The first n coefficients of g as determined by (131) are identical with the corresponding coefficients as determined by (127). Furthermore, since f and g in (127) are periodic functions of order n, the coefficients λ_{n+1} and μ_{n+1} are determined necessarily as polynomials of the coefficients of lower order. Furthermore, the conditions are identical with those arising from (130) and (131) for λ_{n+1} and μ_{n+1} respectively. Hence the forms (130) and (131) for f and g hold for coefficients λ and μ of all orders *up to and inclusive of* $n + 1$. Hence the product is valid for powers of the variable including the $(n + 1)$th. But

$$h^{-1}\epsilon_1 h \cdot h^{-1}\epsilon_1^{-1}h = 1.$$

Hence, necessarily,

$$g[f(z)] = z + (\text{powers of } z > n + 1). \tag{132}$$

Obviously (132) is inconsistent with (127), but results necessarily from the assumption that K_1 in (43) $= 1$, that f and g are of period n and that conditions (125) hold.

Hence $z + z^{n+1}$ cannot be factored into transformations of period n. Hence $P(\lambda)$ in (95) and (123) must $= 0$ identically, and all the other conditions for a necessary relation among the K's must be satisfied identically. Hence

If $z + K_{n+1}z^{n+1} + \cdots$ is to be factored into two transformations of period n, we have as a necessary condition

$$K_{n+1} = 0.$$

24. Suppose now in the discussion preceding Section 18, we choose $t = 2n + 1$. Then t is *odd*. Furthermore, only the coefficients of λ's and μ's of order $n + 1$ in (74), (82) and (76) will be affected. All coefficients λ and μ between the orders $n + 1$ and $2n + 1$ remain independent. The method of Sections 17 and 19 then holds valid so long as

$$k < \frac{t - 1}{2}, \qquad \text{or} \qquad k < n,$$

yielding the necessary conditions

$$K_2 = K_3 = \cdots = K_n = 0.$$

From the previous section this requires in any case

$$K_{n+1} = 0,$$

determining μ_{n+1} in terms of λ's of corresponding and lower orders, hence in terms of λ's of order $< n + 1$. The resulting relation among the K's becomes

$$K_{2n+1} + H_2 K_{2n} + \cdots + H_n K_{n+2} = P(\lambda), \tag{133}$$

where $P(\lambda)$ is a polynomial in λ's of order $< n + 1$, replacing λ_{n+1} by its equivalent in terms of λ's of lower order. Hence if

$$K_{n+2} = K_{n+3} = \cdots = K_{2n} = 0,$$

and $P(\lambda)$ vanishes identically, we obtain the necessary condition

$$K_{2n+1} = 0. \tag{134}$$

This presupposes, too, that the λ-polynomials occurring in the process of obtaining (133) all vanish identically. Hence if

$$K_2 = K_3 = \cdots = K_{2n} = 0, \tag{135}$$

then $K_{2n+1} = 0$.

Following identically the same reasoning as above we have for $t = 3n + 1$, if (135) hold true,

$$K_t + H_2 K_{t-1} + \cdots + H_n K_{t-n+1} = P(\lambda), \tag{136}$$

where $P(\lambda)$ is some polynomial in λ's of order $< t - n$. Hence again, if $P(\lambda) = 0$ and $K_{t-n+1} = \cdots = K_{t-1} = 0$, we must have $K_t = 0$.

The reasoning is clearly general, and we may put $t = sn + 1$ in (136). Hence if

$$K_2 = K_3 = \cdots = K_{sn} = 0, \tag{137}$$

we must have as a necessary condition

$$K_{sn+1} = 0. \tag{138}$$

Furthermore, we observe that the reasoning of Section 23 is general, and that if (137) hold true, then (138) must be satisfied. This is, furthermore, the only way in which a necessary condition may be required among the coefficients K.

We may now state the complete

THEOREM VI. *If the transformation*

$$F(z) = K_1 z + K_2 z^2 + \cdots$$

is to be factorable into two periodic transformations, we must have

$$|K_1| = 1,$$

with a commensurable argument; if $K_1 \neq 1$, the factorization is always possible, the period being so taken that K_1 = the product of the leading coeffi-

cients of the factor transformations; if $K_1 = 1$, the periods of the factor transformations, if they exist, must be equal; if $K_1 = 1$, and $K_2 \neq 0$, then $F(z)$ is always factorable into two transformations of arbitrary period > 2; if $K_1 = 1$ and

$$K_2 = K_3 = \cdots = K_r = 0; \qquad K_{r+1} \neq 0,$$

then $F(z)$ can not be factored into transformations of order r or any factor of r, but can always be factored into transformations of any other order > 2.

25. For example, if

$$F(z) = z + z^{13},$$

$F(z)$ cannot be factored into transformation of period 2,* 3, 4, 6 or 12. It can, however, be factored into transformations of period 5, 7, 8, 9, 10, 11, or any period > 12. On the other hand,

$$F(z) = -z + z^{13}$$

can be factored as the product of transformations of any even period > 2.

It may be observed, also, that in any case where factorization is possible, transformations with equal irreducible periods may be chosen. This is evident from the single restriction (48).

All transformations

$$F(z) = \epsilon z + K_2 z^2 + K_3 z^3 + \cdots; \qquad \epsilon^n = 1$$

are factorable into periodic transformations and constitute a group, the group generated by all periodic transformations. The class defined by $\epsilon = 1$ constitutes a subgroup. The class defined by $\epsilon = 1$, $K_2 = 0$ is a subgroup of the last. The class defined by $\epsilon = 1$, $K_2 = K_3 = 0$ is a subgroup of the previous, and so on. In any of the previous cases, the class defined by $\epsilon = 1$,

$$K_2 = K_3 = \cdots = K_r = 0 \qquad \text{and} \qquad K_{r+1} \neq 0$$

constitutes a subgroup. The latter may be characterized by the index of the highest period of transformations (r) into which $F(z)$ cannot be factored.

It should be observed further that, though

$$F(z) = z + K_{r+1} z^{r+1} + \cdots; \qquad r > 2 \tag{139}$$

cannot be factored into *two* transformations of period r, it can always be factored into *three* transformations of period r. Further, (139) can always be factored into a rotation through a rational angle and two periodic transformations. The latter, too, can be taken of period r, by choosing the corresponding angle.

* See Kasner, loc. cit.

REVERSE CONFORMAL TRANSFORMATIONS (CONFORMAL SYMMETRIES).

26. Kasner also considers another type of conformal transformation. This he calls the *reverse* or *improper* type of transformation, which he defines by

$$y = f(z_0), \tag{140}$$

where f is non-singular at the origin and z_0 is the conjugate of the variable z. He defines reverse transformations of period 2 as *conformal symmetries* since they are conformally reducible to the form $y = z_0$ (conformally equivalent to Schwarzian reflection) and discusses them in parallel with what he terms the *direct conformal transformations.*

We shall now show that no other types of reverse conformal transformations of regular period exist; in other words, every periodic reverse conformal transformation is of irreducible period 2, that is, a conformal symmetry.

27. Denoting the transform of z, $f(z_0)$, by $F(z)$, so that

$$F(z) = f(z_0), \tag{141}$$

we have

$$F_2(z) = f[f(z_0)]_0,$$

where the zero subscript denotes conjugate values.

We now observe that, in general,

$$\begin{gathered}(x + y)_0 = x_0 + y_0, \quad (xy)_0 = x_0y_0, \quad (z^n)_0 = (z_0)^n, \quad (az^n)_0 = a_0z_0^n, \\ [f(z)]_0 = f_0(z_0), \quad (z_0)_0 = z, \quad [f(z_0)]_0 = f_0(z),\end{gathered} \tag{142}$$

where the coefficients of f_0 are the conjugates of those of f.

Hence

$$F_2(z) = f[f_0(z)]. \tag{143}$$

Similarly,

$$\begin{aligned} F_3(z) &= f\{f[f_0(z)]\}_0 \\ &= f\{f_0[f_0(z)]_0\} \\ &= f\{f_0[f(z_0)]\}. \end{aligned}$$

Hence we may write

$$F_{2k-1}(z) = (ff_0)^{k-1}f(z_0), \tag{144}$$

$$F_{2k}(z) = (ff_0)^k(z). \tag{145}$$

28. We shall consider the case $F_{2k-1}(z) = z$ first. If f and f_0 be defined by

$$\begin{aligned} f(z) &= a_1z + a_2z^2 + \cdots, \\ f_0(z) &= b_1z + b_2z^2 + \cdots, \end{aligned} \tag{146}$$

we must have

$$b_r = \frac{|a_r|^2}{a_r}, \qquad |a_1| = 1. \tag{147}$$

Hence $a_1 b_1 = 1$, and

$$f[f_0(z)] = z + \text{(higher powers)}. \tag{148}$$

Hence from (144) we would have

$$a_1 z_0 + \text{(higher powers of } z_0) = z, \tag{149}$$

for all values of z. On comparing coefficients of real and imaginary components of the variable z, the relation (149) is manifestly impossible. Hence, if

$$F_n(z) = z \tag{150}$$

identically, n must be *even.*

29. Suppose now $F_{2k}(z)$ is identically z in (145). Then

$$(ff_0)^k(z) = z. \tag{151}$$

Here ff_0 is a definite transformation. Putting $g(z) = ff_0(z)$, we have then g a *direct* conformal transformation of period k, and we may write

$$g(z) = ff_0(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots; \qquad \epsilon^k = 1. \tag{152}$$

But by (148) the leading coefficient of $ff_0(z)$ is 1. Hence in (152) we must have $\epsilon = 1$. Hence, by Theorem I,

$$\lambda_2 = \lambda_3 = \cdots = 0$$

and we have

$$F_2(z) = f[f_0(z)] = z \tag{153}$$

necessarily. Hence in any case F is of period 2. Hence we may state

THEOREM VII. *Every periodic reverse conformal transformation is of period 2; in other words, the only kind of periodic reverse conformal transformations are conformal symmetries.*

30. In his discussion of conformal symmetries, Kasner obtains the result that the transformation

$$K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \qquad |K_1| = 1, \tag{154}$$

is always factorable into two symmetries if the angle of K_1 is *irrational.* From the discussion in this paper, (154) is factorable into two direct periodic conformal transformations if the angle of K_1 is *rational.* Hence we have

THEOREM VIII. *The transformation*

$$K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \qquad |K_1| = 1,$$

is always factorable either into two conformal symmetries or into two direct periodic conformal transformations.

It would seem also that, in general, a direct periodic conformal transformation is not factorable into two symmetries.

NEW YORK CITY,
August, 1920.

A PRIMARY CLASSIFICATION OF PROJECTIVE TRANSFORMATIONS IN FUNCTION SPACE.*

BY L. L. DINES.

In an earlier paper† we have defined and studied the projective functional transformation

$$(1) \qquad \phi'(x) = \frac{\alpha(x) + \beta(x)\phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy}{\delta + \int_0^1 \epsilon(y)\phi(y)dy},$$

represented symbolically by the array of coefficients

$$\begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}.$$

Obviously the transformation (1) includes as a special case the Fredholm transformation

$$\phi'(x) = \phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy.$$

It also includes other transformations which may properly be said to be *projectively equivalent* to Fredholm transformations. One of the objects of the present paper is to characterize this class of transformations.

Such projective transformations as do not fall into the above class are themselves divided into three classes on the basis of projective equivalence. The classification is accomplished by a consideration of invariant elements of two kinds: *points* and *lineoids,* the lineoid being the natural dual of the point in function space.

§ 1. HOMOGENEOUS COÖRDINATES.

By virtue of its form, (1) transforms every point $\phi(x)$ of function space $\mathfrak{S}_1$ into a point $\phi'(x)$ of the same space, with the exception of those points defined by the equation

$$\delta + \int_0^1 \epsilon(y)\phi(y)dy = 0.$$

With a view to attaining a domain which is closed under the projective transformation we introduce homogeneous coördinates. To this end, we make in (1) the substitutions

$$(2) \qquad \phi(x) = \omega_1(x)/\omega_2, \qquad \phi'(x) = \omega_1'(x)/\omega_2',$$

* Read under slightly different title before the American Mathematical Society, September 3, 1919.

† *Transactions of the American Mathematical Society,* Vol. 20, pages 45–65. This paper will be referred to as *Proj. Trans.*

denote this range by $\mathfrak{I}_1$; that is

$$\mathfrak{I}_1 \equiv [\textit{all real numbers from } 0 \textit{ to } 1, \textit{ inclusive}].$$

Let us likewise define

$$\mathfrak{I}_2 \equiv [\textit{all real numbers from } 1 \textit{ to } 2, \textit{ including } 2],$$
$$\mathfrak{I}_1 + \mathfrak{I}_2 \equiv [\textit{all real numbers from } 0 \textit{ to } 2, \textit{ inclusive}].$$

By composition of these ranges we get the following composite ranges (each represented geometrically by the points of a square):

$$\mathfrak{I}_1\mathfrak{I}_1, \quad \mathfrak{I}_1\mathfrak{I}_2, \quad \mathfrak{I}_2\mathfrak{I}_1, \quad \mathfrak{I}_2\mathfrak{I}_2, \quad (\mathfrak{I}_1 + \mathfrak{I}_2)(\mathfrak{I}_1 + \mathfrak{I}_2).$$

Now let us suppose the binary elements $(\omega_1(x), \omega_2)$ and $(\omega_1'(x), \omega_2')$ to be replaced respectively by the binary elements $(\omega_1(x), \omega_2(x))$ and $(\omega_1'(x), \omega_2'(x))$, with the understanding that $\omega_2(x)$ and $\omega_2'(x)$ are constant functions having the definitions:

$$\omega_2(x) = \omega_2, \qquad \omega_2'(x) = \omega_2', \qquad x \textit{ on } \mathfrak{I}_2.$$

Then the equations of (3) are equivalent to

$$(6) \qquad \begin{aligned} r\omega_1'(x) &= \beta(x)\omega_1(x) + \int_0^1\gamma(xy)\omega_1(y)dy + \int_1^2\alpha(x)\omega_2(y)dy, \\ r\omega_2'(x) &= \qquad\omega_2(x) + \int_0^1 \epsilon(y)\omega_1(y)dy + \int_1^2(\delta - 1)\omega_2(y)dy, \end{aligned}$$

and by the classic method of Fredholm* for reducing a system of integral equation to a single equation, this pair of equations can be represented by

$$(7) \qquad r\rho(x)\omega'(x) = \omega(x) + \int_0^2\kappa(xy)\omega(y)dy, \qquad x \textit{ on } \mathfrak{I}_1 + \mathfrak{I}_2,$$

where

$$\omega(x) = \begin{cases} \omega_1(x) & \textit{on } \mathfrak{I}_1 \\ \omega_2 & \textit{on } \mathfrak{I}_2 \end{cases} \qquad \omega'(x) = \begin{cases} \omega_1'(x) & \textit{on } \mathfrak{I}_1 \\ \omega_2' & \textit{on } \mathfrak{I}_2 \end{cases} \qquad \rho(x) = \begin{cases} 1/\beta(x) & \textit{on } \mathfrak{I}_1 \\ 1 & \textit{on } \mathfrak{I}_2 \end{cases}$$

$$\kappa(x, y) = \begin{cases} \gamma(x, y)/\beta(x) & \textit{on } \mathfrak{I}_1\mathfrak{I}_1 \\ \alpha(x)/\beta(x) & \textit{on } \mathfrak{I}_1\mathfrak{I}_2 \\ \epsilon(y) & \textit{on } \mathfrak{I}_2\mathfrak{I}_1 \\ \delta - 1 & \textit{on } \mathfrak{I}_2\mathfrak{I}_2 \end{cases}$$

The transformation (7) is an ordinary Fredholm transformation, transforming a function $\omega(x)$ on $\mathfrak{I}_1 + \mathfrak{I}_2$ into a function $r\rho(x)\omega'(x)$ on the same interval. The kernel $\kappa(xy)$ is continuous on the square $(\mathfrak{I}_1 + \mathfrak{I}_2)(\mathfrak{I}_1 + \mathfrak{I}_2)$ except for the lines of possible discontinuity $x = 1$ and $y = 1$.

In a similar manner, the lineoid transformation (5) can be replaced by

$$(7') \qquad r\rho'(y)v'(y) = v(y) + \int_0^2 v(x)\kappa'(x, y)dx, \qquad y \textit{ on } \mathfrak{I}_1 + \mathfrak{I}_2,$$

* Acta mathematica, Vol. 27.

where

$$v(y) = \begin{cases} v_1(y) & on\, \mathfrak{J}_1 \\ v_2 & on\, \mathfrak{J}_2 \end{cases}$$

$$v'(y) = \begin{cases} v_1(y) & on\, \mathfrak{J}_1 \\ v_2' & on\, \mathfrak{J}_2 \end{cases}$$

$$\rho'(y) = \begin{cases} 1/\beta'(y) & on\, \mathfrak{J}_1 \\ 1 & on\, \mathfrak{J}_2 \end{cases}$$

$$\kappa'(x, y) = \begin{cases} \gamma'(x, y)/\beta'(y) & on\, \mathfrak{J}_1\mathfrak{J}_1 \\ \epsilon'(y)/\beta'(y) & on\, \mathfrak{J}_2\mathfrak{J}_1 \\ \alpha'(x) & on\, \mathfrak{J}_1\mathfrak{J}_2 \\ \delta' - 1 & on\, \mathfrak{J}_2\mathfrak{J}_2 \end{cases}$$

The kernels $\kappa(x, y)$ and $\kappa'(x, y)$ of the transformations (7) and (7′) satisfy the two identities

$$\kappa(x, y) + \kappa'(x, y) + \int_0^2 \kappa(x, z)\kappa'(z, y)dz = 0$$
$$\kappa'(x, y) + \kappa(x, y) + \int_0^2 \kappa'(x, z)\kappa(z, y)dz = 0$$

on the square $(\mathfrak{J}_1 + \mathfrak{J}_2)(\mathfrak{J}_1 + \mathfrak{J}_2)$, as may be verified by use of the relations (9) and (9′) of *Proj. Trans.* That is, they are *reciprocal kernels* in the Fredholm sense.

§ 4. Invariant Points and Lineoids.

We now inquire as to what points, if any, are transformed into themselves by (3), restricting ourselves for the present however to the important special case in which

$$\beta(x) = 1. \tag{8}$$

From (7) which is equivalent to (3), we see that under the condition (8), the equation for such invariant points is

$$(1 - r)\omega(x) + \int_0^2 \kappa(x, y)\omega(y)dy = 0, \qquad x \text{ on } \mathfrak{J}_1 + \mathfrak{J}_2. \tag{9}$$

Every non-zero solution $\omega(x)$ of this equation determines a binary element $(\omega_1(x), \omega_2(x))$ in which

$$\omega_1(x) = \omega(x) \qquad on\, \mathfrak{J}_1,$$
$$\omega_2(x) = \omega(x) \qquad on\, \mathfrak{J}_2.$$

If $\omega_2(x)$ is constant, this binary element represents an invariant point. And $\omega_2(x)$ cannot be other than a constant except in the single case $r = 1$, as will be evident upon consideration of the content of equation (9) when x is on $\mathfrak{J}_2$, namely

$$(1 - r)\omega_2(x) + \int_0^1 \epsilon(y)\omega_1(y)dy + \int_0^2 (\delta - 1)\omega_2(y)dy = 0. \tag{10}$$

In case $r = 1$, a solution $\omega = (\omega_1(x), \bar{\omega}_2(x))$ may exist in which $\bar{\omega}_2(x)$ is not constant, but corresponding to every such solution there will be a solution $\omega = (\omega_1(x), \omega_2)$ in which $\omega_2 = \int_0^1 \bar{\omega}_2(y)dy$. This solution will represent an invariant point unless $\omega_1(x) = \omega_2 = 0$.

Equation (9) is an homogeneous Fredholm integral equation containing the parameter r. The non-zero solutions $\omega(x)$ corresponding to values of r different from 1 are the so-called *fundamental functions in x* for the kernel $\kappa(x, y)$. If we denote by the term *asymptotic fundamental functions* those non-zero solutions $\omega(x)$ corresponding to the parameter value $r = 1$, we may state our conclusions in the form.

THEOREM I: *If $\beta(x) = 1$, the invariant points $\omega(x) = (\omega_1(x), \omega_2)$ of the transformation* (3) *are precisely the ordinary fundamental functions in x of the kernel $\kappa(x, y)$, together with those asymptotic fundamental functions in x which are constant on the interval $\mathfrak{J}_2$.*

By an analogous consideration of (7′), we obtain

THEOREM I′: *If $\beta(x) = 1$, the invariant lineoids $v(y) = (v_1(y), v_2)$ of the transformation* (3) *are precisely the ordinary fundamental functions in y of the kernel $\kappa'(x, y)$, together with those asymptotic fundamental functions in y which are constant on the interval $\mathfrak{J}_2$.*

In this theorem the invariant lineoids are characterized by means of the reciprocal kernel $\kappa'(x, y)$. They are characterized directly in terms of $\kappa(x, y)$ in the following

COROLLARY: *If $\beta(x) = 1$, the invariant lineoids of the transformation* (3) *are precisely the ordinary fundamental functions in y of the kernel $k(x, y)$, together with those asymptotic fundamental functions which are constant on the interval $\mathfrak{J}_2$.*

To prove this we need only show that the solutions of the two equations

$$(1 - r')v(y) + \int_0^2 v(x)\kappa'(x, y)\,dx = 0 \tag{11'}$$

and

$$(1 - r)v(y) + \int_0^2 v(x)\kappa(x, y)\,dx = 0 \tag{11}$$

coincide.

Suppose $\bar{v}(y)$ is a solution of (11′). Then for some value of r',

$$r'\bar{v}(y) = \bar{v}(y) + \int_0^2 \bar{v}(x)\kappa'(x, y)\,dx. \tag{12}$$

We multiply equation (11) by r' and write it in the form

$$r'rv(y) = r'v(y) + r'\int_0^2 v(x)\kappa(x, y)\,dx. \tag{13}$$

Upon substituting in (13) $\bar{v}$ for v in the left member and the value of $r'\bar{v}$ from (12) for $r'v$ in the right member, we obtain after some rearrangement

$$r'r\bar{v}(y) = \bar{v}(y) + \int_0^2 \bar{v}(x)\Big[\kappa'(x, y) + \kappa(x, y) + \int_0^2 \kappa'(x, z)\kappa(z, y)\,dz\Big]dx.$$

Since $\kappa(x, y)$ and $\kappa'(x, y)$ are reciprocal kernels this reduces to

$$r'r\bar{v}(y) = \bar{v}(y),$$

which proves that $\bar{v}(y)$ is a solution of (11) corresponding to the parameter value $r = 1/r'$.

In a similar manner it can be shown that each solution of (11) is a solution of (11′).

§ 5. SYMMETRIC PROJECTIVE TRANSFORMATIONS.

The theorems of the preceding section suggest interesting properties of those projective transformations for which the corresponding kernels $\kappa(x, y)$ are symmetric. The most general such transformation is of form

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2, \\ r\omega_2' &= \int_0^1 \alpha(y)\omega_1(y)dy + \delta\omega_2, \end{aligned}$$

where $\gamma(x, y) = \gamma(y, x)$.

Transformations of this form will be called *symmetric projective transformations*.

From the well-known theory of integral equations with symmetric kernels,* we have by virtue of the theorems of the preceding section

THEOREM II. *Every symmetric projective transformation has at least one invariant point, and one invariant lineoid, which are not conjoint.*

The existence of the invariant elements follows from the fact that every symmetric kernel has a characteristic value; the non-conjointness from the fact that every such characteristic value is a simple pole of the resolvent, and hence (see Goursat, loc. cit., page 411) for every fundamental function ω in x of the kernel $\kappa(x, y)$ there is a fundamental function v in y such that

$$\int_0^2 v(y)\omega(y)dy \neq 0,$$

that is

$$\int_0^1 v_1(y)\omega_1(y)dy + v_2\omega_2 \neq 0.$$

§ 6. PROJECTIVE TRANSFORMATIONS WITHOUT INVARIANT POINTS AND LINEOIDS.

While, as we have just seen, every symmetric projective transformation admits an invariant point and lineoid, such is not the case with non-symmetric transformations. In this section we present four examples of transformations qualified respectively as follows:

1. Admitting an invariant lineoid, but no invariant point.
2. Admitting an invariant point, but no invariant lineoid.
3. Admitting neither invariant point nor lineoid.
4. Admitting an invariant point and an invariant lineoid, but no pair which are not conjoint.

* See, for instance, Goursat's *Cours d'analyse mathematique*, Vol. III, § 587.

We shall make use of the following

LEMMA:* If the traces

$$A_n = \int_0^2 \int_0^2 \cdots \int_0^2 \kappa(s_1, s_2)\kappa(s_2, s_3) \cdots \kappa(s_n, s_1)ds_1 ds_2 \cdots ds_n$$

of the kernel $\kappa(x, y)$ are zero for all values of n greater than or equal to 3, then $\kappa(x, y)$ admits no characteristic value, and hence no ordinary fundamental function.

We take now any complete† normalized orthogonal system of continuous functions ϕ_n, $n = 1, 2, \cdots$, on the interval $\mathfrak{J}_1$.

From this system which is orthogonal on $\mathfrak{J}_1$ we form a system Φ_n, $n = 0, 1, 2, \cdots$, orthogonal on $\mathfrak{J}_1 + \mathfrak{J}_2$, as follows:

$$\Phi_0 \equiv [0 \text{ on } \mathfrak{J}_1, 1 \text{ on } \mathfrak{J}_2],$$
$$\Phi_n \equiv [\phi_n \text{ on } \mathfrak{J}_1, 0 \text{ on } \mathfrak{J}_2], \qquad n = 1, 2, \cdots.$$

Our examples will be constructed in terms of the functions $\{\phi_n\}$; and in each case the kernel $\kappa(x, y)$ to which the transformation gives rise will be expansible in terms of the functions $\{\Phi_n\}$.

Example 1. Consider the transformation

$$\begin{pmatrix} 1+\gamma & \alpha \\ 0 & 1 \end{pmatrix} \tag{14}$$

in which

$$\gamma(x, y) \equiv \sum_{n=1}^{\infty} c_n \phi_{n+1}(x)\phi_n(y), \qquad \alpha(x) \equiv \phi_1(x),$$

the coefficients c_n being all distinct from zero and so chosen that the series for $\gamma(x, y)$ converges uniformly on the square $\mathfrak{J}_1\mathfrak{J}_1$.

The corresponding kernel $\kappa(x, y)$ can be written in the form

$$\kappa(x, y) \equiv \sum_{n=0}^{\infty} c_n \Phi_{n+1}(x)\Phi_n(y) \quad \text{on } (\mathfrak{J}_1 + \mathfrak{J}_2)(\mathfrak{J}_1 + \mathfrak{J}_2),$$

where $c_0 = 1$.

The traces of this kernel $\kappa(x, y)$ are all zero. Hence by the lemma the kernel admits no singular values, and no ordinary fundamental functions.

Nor does it admit any asymptotic fundamental functions which can furnish invariant points. For the equation

$$\int_0^2 \kappa(x, y)\omega(y)dy = 0$$

is equivalent to

$$\sum_{n=0}^{\infty} c_n \Phi_{n+1}(x) \int_0^2 \Phi_n(y)\omega(y)dy = 0.$$

* Cf. Goursat, loc. cit., page 428.

† An orthogonal system of functions is said to be *complete* if there is no function which is orthogonal to all the functions of the system, and such that the integral of its square is 1. Cf. Goursat, loc. cit., pages 445 and 446.

Multiplying this equation by $\Phi_{k+1}(x)$ and integrating with respect to x, we get, on account of the orthogonality,

$$c_k \int_0^2 \Phi_k(y)\omega(y)dy = 0, \qquad k = 0, 1, 2, \cdots.$$

That is, since $c_k \neq 0$, $\omega(y)$ must be orthogonal to all the functions $\Phi_k(y)$, on the interval $\mathfrak{J}_1 + \mathfrak{J}_2$.

But from the definitions of the Φ's, it is clear that there can be no such function ω which is constant on $\mathfrak{J}_2$. Hence, *the transformation*

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2, \\ r\omega_2' &= \omega_2, \end{aligned}$$

represented by (14), *admits no invariant point. It admits the invariant lineoid* $\omega_2 = 0$.

Example 2. *The transformation*

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy, \\ r\omega_2' &= \omega_2 \quad + \int_0^1 \epsilon(y)\omega_1(y)dy, \end{aligned}$$

in which

$$\gamma(x, y) \equiv \sum_{n=1}^{\infty} c_n\phi_n(x)\phi_{n+1}(y), \qquad \epsilon(y) \equiv \phi_1(y),$$

has no invariant lineoid, but admits the invariant point (0, 1).

The kernel $\kappa(x, y)$ for this transformation can be written in the form

$$\kappa(x, y) = \sum_{n=0}^{\infty} c_n\Phi_n(x)\Phi_{n+1}(y),$$

from which fact the properties of the transformation follow as in Example 1.

Example 3. Consider the transformation whose coefficients are*

$$\begin{aligned} \gamma(x, y) &\equiv \sum_{n=1}^{\infty} c_n[\phi_{2n+1}(x)\phi_{2n-1}(y) + \phi_{2n}(x)\phi_{2n+2}(y)], \\ \alpha(x) &\equiv \phi_1(x), \qquad \epsilon(y) \equiv \phi_2(y), \qquad \delta \equiv 1. \end{aligned} \tag{15}$$

The corresponding kernel $\kappa(x, y)$ can be written in the form

$$\kappa(x, y) = \Phi_1(x)\Phi_0(y) + \Phi_0(x)\Phi_2(y)$$

$$\sum_{n=1}^{\infty} c_n[\Phi_{2n+1}(x)\Phi_{2n-1}(y) + \Phi_{2n}(x)\Phi_{2n+2}(y)].$$

All the traces of this kernel are zero; hence it has no ordinary fundamental functions. Nor does it admit asymptotic fundamental functions in either x or y which are constant on $\mathfrak{J}_2$, as may be shown by the method used in Example 1. Hence *the transformation defined by* (15) *admits no invariant element.*

* The form of this transformation was suggested by an example kindly furnished me by Professor E. W. Chittenden.

Example 4. Consider the transformation

$$r\omega_1'(x) = \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2,$$
$$r\omega_2' = \omega_2,$$

where

$$\gamma(x, y) \equiv \sum_{n=2}^{\infty} c_n\phi_{n-1}(x)\phi_n(y), \qquad \alpha(x) \equiv \phi_1(x).$$

The corresponding kernel $\kappa(x, y)$ given by

$$\kappa(x, y) = \Phi_1(x)\Phi_0(y) + \sum_{n=2}^{\infty} c_n\Phi_{n+1}(x)\Phi_n(y)$$

has no ordinary fundamental functions since all of its traces are zero.

The transformation admits a single invariant point given by the asymptotic fundamental function

$$\omega(x) = \Phi_1(x) = (\phi_1(x), 0),$$

while it admits the two invariant lineoids

$$v(y) = \Phi_0(y) = (0, 1), \qquad \iota(y) = \Phi_2(y) = (\phi_2(y), 0).$$

It can be verified immediately that *the invariant point is on both of the invariant lineoids.*

§ 7. REDUCTION TO CANONICAL FORM.

Two transformations

$$\bar{S} \equiv \begin{pmatrix} \bar{\beta} + \bar{\gamma} & \bar{\alpha} \\ \bar{\epsilon} & \bar{\delta} \end{pmatrix}, \qquad \bar{\bar{S}} \equiv \begin{pmatrix} \bar{\bar{\beta}} + \bar{\bar{\gamma}} & \bar{\bar{\alpha}} \\ \bar{\bar{\epsilon}} & \bar{\bar{\delta}} \end{pmatrix}$$

are said to be projectively equivalent if there exists a non-singular projective transformation T with inverse T^{-1}:

$$T \equiv \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}, \qquad T^{-1} \equiv \begin{pmatrix} \beta' + \gamma' & \alpha' \\ \epsilon' & \delta' \end{pmatrix},$$

such that

$$T\bar{S}T^{-1} = \bar{\bar{S}}. \tag{16}$$

The relation between $\bar{S}$ and $\bar{\bar{S}}$ is evidently a reciprocal one, since from (16) it follows that $T^{-1}\bar{\bar{S}}T = \bar{S}$.

Suppose now that $\bar{S}$ admits the invariant point $\bar{\omega} \equiv (\bar{\omega}_1(x), \bar{\omega}_2)$. Then $\bar{\bar{S}}$ admits as invariant point the transform of $\bar{\omega}$ by T. For from the equation

$$\bar{S}\bar{\omega} = r\bar{\omega},$$

which expresses the fact that $\bar{\omega}$ is invariant under $\bar{S}$, there follows by use

of (16)

$$
\begin{aligned}
\bar{\bar{S}}(T\bar{\omega}) &= T\bar{S}T^{-1}(T\bar{\omega}) \\
&= T\bar{S}\bar{\omega} \\
&= Tr\bar{\omega} \\
&= r(T\bar{\omega}),
\end{aligned}
$$

which expresses the fact that $T\bar{\omega}$ is invariant under $\bar{\bar{S}}$.

Likewise, if $\bar{S}$ admits the invariant lineoid $\bar{v}$, then $\bar{\bar{S}}$ admits as invariant lineoid the transform of $\bar{v}$ by T.

It is evident from these remarks that a *necessary* condition for projective equivalence is a one-to-one correspondence between the invariant elements of two transformations. This condition is not sufficient, and we shall not undertake here the general question of sufficient conditions.

We shall show, however, that a projective transformation which admits *any* invariant element is projectively equivalent to a transformation of one of the three forms

$$
\begin{pmatrix} \beta+\gamma & 0 \\ \epsilon & \delta \end{pmatrix}, \quad \begin{pmatrix} \beta+\gamma & \alpha \\ 0 & \delta \end{pmatrix}, \quad \begin{pmatrix} \beta+\gamma & 0 \\ 0 & \delta \end{pmatrix}, \tag{17}
$$

of which the first admits the invariant point (0, 1), the second admits the invariant lineoid (0, 1), while the third admits both of these invariant elements.

We shall need to make use of the explicit forms of the coefficients of $\bar{\bar{S}}$ in terms of the coefficients of T, $\bar{S}$, and T^{-1} as determined by (16). These can easily be written down by use of the formulas obtained in *Proj. Trans.*, § 5. In terms of the abbreviated notations there used, we have

$$
\begin{aligned}
\bar{\bar{\beta}} &= \beta\bar{\beta}\beta' = \bar{\beta}, \\
\bar{\bar{\gamma}} &= \beta\bar{\beta}\gamma' + \beta\bar{\gamma}\beta' + \gamma\bar{\beta}\beta' + (J\gamma\bar{\gamma})\beta' + \alpha\cdot\bar{\epsilon}\beta' + \beta J\bar{\gamma}\gamma' \\
&\qquad + J\gamma\bar{\beta}\gamma' + JJ\gamma\bar{\gamma}\gamma' + \alpha J\bar{\epsilon}\gamma' + \beta J\bar{\alpha}\epsilon' + JJ\gamma\bar{\alpha}\epsilon' + J\alpha\bar{\delta}\epsilon', \\
\bar{\bar{\alpha}} &= \beta(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + J\gamma(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + \alpha(J\bar{\epsilon}\alpha' + \bar{\delta}\delta'), \\
\bar{\bar{\epsilon}} &= (\epsilon\bar{\beta} + J\epsilon\bar{\gamma} + \delta\bar{\epsilon})\beta' + J(\epsilon\bar{\beta} + J\epsilon\bar{\gamma} + \delta\bar{\epsilon})\gamma' + (J\epsilon\bar{\alpha} + \delta\bar{\delta})\epsilon', \\
\bar{\bar{\delta}} &= J\epsilon(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + (J\epsilon\bar{\alpha} + \delta\bar{\delta})\delta'.
\end{aligned} \tag{18}
$$

Suppose now that $\bar{S}$ admits the invariant point $(\bar{\omega}_1, \bar{\omega}_2)$, that is

$$
\begin{aligned}
r\bar{\omega}_1 &= \bar{\beta}\bar{\omega}_1 + J\bar{\gamma}\bar{\omega}_1 + \bar{\alpha}\bar{\omega}_2, \\
r\bar{\omega}_2 &= \qquad\quad\;\; J\bar{\epsilon}\bar{\omega}_1 + \bar{\delta}\bar{\omega}_2.
\end{aligned} \tag{19}
$$

Then if T be so chosen that

$$
(\alpha', \delta') \equiv (\bar{\omega}_1, \bar{\omega}_2),
$$

the value of $\bar{\bar{\alpha}}$ as determined by (18) may in view of (19) be written

$$
\bar{\bar{\alpha}} = r(\beta\alpha' + J\gamma\alpha' + \alpha\delta').
$$

Hence

$$\bar{\bar{\alpha}} = 0,$$

on account of the relations between the coefficients of T and T^{-1} (see (9′) of *Proj. Trans.*, I); and the transformation $\bar{\bar{S}}$ projectively equivalent to $\bar{S}$ is of the form occurring first in (17).

Similarly, it may be seen that if $\bar{S}$ admits the invariant lineoid $(\bar{v}_1, \bar{v}_2)$, and if T be so chosen that

$$(\epsilon, \delta) \equiv (\bar{v}_1, \bar{v}_2),$$

then

$$\bar{\bar{\epsilon}} = 0;$$

that is, the transformation $\bar{\bar{S}}$ is of the form appearing second in (17).

The third transformation in (17) admits the invariant point (0, 1) and the invariant lineoid (0, 1)—two elements which are not conjoint. Hence no transformation can be equivalent to it unless it admits two similarly related invariant elements. Let us then suppose that the transformation $\bar{S}$ admits the invariant point $(\bar{\omega}_1, \bar{\omega}_2)$ and the invariant lineoid $(\bar{v}_1, \bar{v}_2)$, and that these two elements are not conjoint, that is

$$J\bar{\omega}_1\bar{v}_1 + \bar{\omega}_2\bar{v}_2 = 1. \tag{20}$$

We shall show that under these conditions, T may be so chosen that $\bar{\bar{S}}$ is of the form occurring third in (17). For this purpose it will be sufficient, in view of the two cases just preceding, to exhibit a transformation T with inverse T^{-1} satisfying the *two* conditions

$$(\epsilon, \delta) \equiv (\bar{v}_1, \bar{v}_2), \qquad (\alpha', \delta') \equiv (\bar{\omega}_1, \bar{\omega}_2).$$

We may on account of (20) and the homogeneity of coördinates assume without loss of generality that *if* $J\bar{\omega}_1\bar{v}_1 = 0$, *then* $\bar{\omega}_2 = \bar{v}_2 = 1$. This will simplify the discussion.

Consider the transformation

$$T = \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}$$

in which

$$\beta \equiv 1, \qquad \gamma \equiv c\bar{\omega}_1 \cdot \bar{v}_1, \qquad \alpha \equiv -\bar{\omega}_1, \qquad \epsilon \equiv \bar{v}_1, \qquad \delta \equiv \bar{v}_2, \tag{21}$$

c being a constant to be determined.

The coefficients α', δ' of the inverse transformation T^{-1} are given by the formulas*

$$\alpha'(x) = A(x)/B, \qquad \delta' = D/B,$$

* *Proj. Trans.*, (8).

§ 9. FREDHOLM TRANSFORMATIONS.

According to Theorem V, every non-singular transformation of Type I is projectively equivalent to the product of a functional multiplication and a transformation of form

$$\begin{pmatrix} 1+\gamma & 0 \\ 0 & 1 \end{pmatrix}.$$

This latter is, in non-homogeneous coördinates, the ordinary Fredholm transformation

$$\phi' = \phi + J\gamma\phi.$$

Conversely, if a projective transformation is projectively equivalent to the product of a functional multiplication and an ordinary Fredholm transformation, it must admit an invariant lineoid and an invariant point not on it. Hence

THEOREM VI: *A necessary and sufficient condition that a non-singular projective transformation be projectively equivalent to the product of a functional multiplication and an ordinary Fredholm transformation is that it admit an invariant lineoid and an invariant point which are not conjoint.*

COROLLARY: *Every non-singular symmetric projective transformation is projectively equivalent to an ordinary Fredholm transformation and multiplication by a constant.*

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A GENERAL THEORY OF LIMITS.

BY E. H. MOORE AND H. L. SMITH.

Introduction.

I. *As to Notions.*—In this paper we investigate a simple general limit of which, as will appear in § 5, the various classical limits of analysis are actually instances.* The general limit in question is an obvious generalization of the following two limits:

1. An infinite sequence $\{a_n\}$ of real or (ordinary) complex numbers a_n $(n = 1, 2, \cdots)$ converges to a number a as a limit, in notation: $L_{n \to \infty} a_n = a$,—as clearly defined about a century ago,—in case for every positive number e there exists a positive integer n_e of such a nature that for every integer $n \geqq n_e$ it is true that in absolute value $a_n - a$ is at most e. Here the numerical sequence $\{a_n\}$ may be considered as a numerically valued function $\alpha \equiv (a_n | n)$ of the positive integer n (or on the range $[n]$ of positive integers n), viz., $\alpha(n) \equiv a_n$ for every n.

2. Relative to a general (i.e., any particular) class $\mathfrak{Q} \equiv [q]$ of general elements q and the class $\mathfrak{S} = [s]$ of all finite classes s of elements q, a numerically valued function $\alpha \equiv (\alpha(s) | s)$ on the range S converges to a number a as limit, in notation: $L_s \alpha(s) = a$, in case for every positive number e there exists a class s_e of such a nature that for every class s including s_e it is true that in absolute value $\alpha(s) - a$ is at most e. The limit (2) belongs to General Analysis, i.e., to that doctrine of analysis in which a general class, here $\mathfrak{Q}$, plays a fundamental rôle. The limit (2), introduced† in 1915 by the senior author, plays a central rôle in his second theory‡ of Linear Integral Equations in General Analysis. This general theory has as notable instances: (*a*) Hilbert's theory of limited quadratic forms in a denumerable infinitude of variables; here the Hilbert space $[\alpha]$ of infinite sequences $\alpha \equiv (\alpha(n) | n)$ of real numbers with convergent $\Sigma_n \alpha(n)\alpha(n)$ plays a central rôle; (*b*) an analogous theory in which the corresponding rôle

* The only other attempt in this direction is by Dimitry-Kryjanowsky, *Nouvelles Annales de Mathématiques*, Ser. 4, Vol. 14 (1914), pp. 49–54. In this paper it is shown how all the classical limits, by means of a transformation, may be reduced to a certain canonical form. The theory is not as general as the present one; indeed it does not include the limit 2) below.

† E. H. Moore, "Definition of Limit in General Integral Analysis," *Proceedings of the National Academy of Sciences*, Vol. 1 (1915), pp. 628–632.

‡ (Addition of Sept. 19, 1922.) The basis of this theory is included in his paper, "On Power Series in General Analysis," *Festschrift David Hilbert zu seinem sechzigsten Geburtstag*, pp. 355–364, Berlin, 1922, and *Mathematische Annalen*, vol. 86, pp. 30–39 (1922).

is played by the Hellinger space $[\alpha]$ of real-valued functions $\alpha \equiv (\alpha(x) | x)$ of the real variable x on the interval $(0 \leqq x \leqq 1)$ with $\alpha(0) = 0$ and $\int_0^1 \frac{d\alpha(x)d\alpha(x)}{d\beta(x)}$ convergent, β being a fixed monotone increasing function of x; (*c*) various instances involving integration over function-spaces.

It is obvious that the limits (1), (2) are in close analogy. In each case a numerically valued function $\alpha = (\alpha(p) | p)$ on a certain range $\mathfrak{P} \equiv [p]$ is said to converge, under certain conditions, to a number a as limit; in (1) $\mathfrak{P} \equiv [n]$, in (2) $\mathfrak{P} \equiv \mathfrak{S} \equiv [s]$; moreover, notations apart, the conditions in (1) become the conditions in (2) on replacing the relation $\geqq$, between two positive integers $(n \geqq n_e)$, by the relation *including* or $\supset$, between two classes $(s \supset s_e)$. Thus the authors were led independently to the general limit (3) of which the limits (1), (2) are special instances.

3. Consider a *general class* $\mathfrak{P} \equiv [p]$ of members or *elements* p and a *binary relation* R on the class $\mathfrak{P}$; according as an element p_1 is or is not in the relation R to an element p_2 write $p_1 \mathrm{R} p_2$ or $p_1 {}^{-}\mathrm{R} p_2$. Then we say that a *numerically valued function* $\alpha \equiv (\alpha(p) | p)$* *on the range* $\mathfrak{P}$ *converges (with respect to the relation* R) *to a number* a *as limit,* in notation:

$$\mathrm{L}_p \alpha(p) = a \qquad \text{or} \qquad \mathrm{L}\alpha = a,$$

or, with the relation R in evidence,

$$\mathrm{L}_{p\mathrm{R}} \alpha(p) = a \qquad \text{or} \qquad \mathrm{L}_{\mathrm{R}}\alpha = a,$$

in case for every positive number e *there exists an element* p_e *of such a nature that for every* $p\mathrm{R}p_e$ (i.e., for every element p in the R relation to p_e) *it is true that in absolute value* $\alpha(p) - a$ *is at most* e.†

In order that the theory of the general limit (3) may include the principal parts of the theories of the limits (1), (2), we impose upon the relation R two conditions obviously satisfied in (1), (2), viz., the conditions: 1) R is *transitive* (R^T),—if p_1 is in the R relation to p_2 and p_2 is in the R relation to p_3, then p_1 is in the R relation to p_3; 2) R has the *composition property* (R^C),—for every two (not necessarily distinct) elements $p_1 p_2$ there exists an element $p_3 \mathrm{R}(p_1, p_2)$, that is, an element p_3 in the R relation to each of the elements $p_1 p_2$.

Thus, as *fundamental system* Σ *of notions* for the general limit (3), we have the system:

$$\Sigma \equiv (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\mathrm{on}\,\mathfrak{P}\mathfrak{P}\cdot TC});$$

* Throughout the paper α will denote a numerically valued function on the range, not restricted to be single-valued unless so stated.

† If α is not single-valued, this inequality is understood to hold for all determinations of $\alpha(p)$; unless otherwise stated similar understandings will hold in the case of all inequalities involving multiply-valued functions.

that is, the class $\mathfrak{A} \equiv [a]$ of all real or (ordinary) complex numbers a, a general class $\mathfrak{P}$, and a binary relation R on the class $\mathfrak{P}$ which is transitive and has the composition property.

II. *As to Scope.*—The present study of the general limit (3) is arranged as follows:

§ 1. Elementary theorems.

§ 2. Necessary and sufficient conditions for the existence of a limit.

§ 3. Some modes of convergence.

§ 4. Quasi-limits. Upper and lower limits.

§ 5. Limits as to norm.

§ 6. Types of uniform convergence as to a general parameter.

§ 7. Double limits.

§ 8. Lemmas: Revised formulation of certain theorems of Fréchet.

§ 9. Composite range. Continuity.

III. *As to Notations.*—In order to expound briefly and luminously the considerable body of doctrine outlined in II, we make systematic use of readily understood notations for constantly recurring logical and mathematical notions.

For instance, the definition in I (3) of $\mathrm{L}\alpha = a$ we write: *there exists a system* $(p_e|e)$ *such that for every* $p\mathrm{R}p_e$ *it is true that* $|\alpha(p) - a| \leqq e$, or even more briefly, *there exists a system* $(p_e|e)$ *such that*

$$|\alpha(p) - a| \leqq e \cdot \qquad (p\mathrm{R}p_e).$$

Throughout e denotes a positive number.

§ 1. Elementary Theorems.

In the Introduction we have indicated the fundamental system

$$\Sigma = (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\mathrm{on}\,\mathfrak{P}\mathfrak{P}\cdot TC})$$

of notions under consideration, and defined the associated limit notion. We here add certain simple explanations and propositions.

It follows from R^C that for every element p there is an element p_p such that $p_p\mathrm{R}p$. For we have only to take in the definition of R^C $p_1 = p$, $p_2 = p$ and then take $p_p = p_3$.

In case $p\mathrm{R}p$ for every p, R is *reflexive*, in notation R^R. It is not assumed that R is reflexive. We shall however define an associated relation R_* which is transitive, has the composition property and is also reflexive. We define: $p_1\mathrm{R}_*p_2$ in case either $p_1\mathrm{R}p_2$ or $p_1 = p_2$. The proof that $\mathrm{R}_*{}^{TCR}$ is simple and is omitted.

To the definition of limit given in the introduction should be added the following one: α *converges to* $\sigma\infty$ $(\sigma = +$ or $-)$ *as limit*, in notation:

$$\mathrm{L}_p\alpha(p) = \sigma\infty \qquad \text{or} \qquad \mathrm{L}\alpha = \sigma\infty,$$

or, with the relation R in evidence,

$$L_{pR}\alpha(p) = \sigma\infty \qquad \text{or} \qquad L_R\alpha = \sigma\infty$$

in case there exists a system $(p_e|e)$ *such that*

$$\sigma\alpha(p) \geqq e \qquad (pRp_e).$$

The limit of α *exists* in case there exists a number a such that $L\alpha = a$. The limit of α *exists finite or infinite* in case either $L\alpha = +\infty$, $L\alpha = -\infty$, or there is some a such that $L\alpha = a$.

By $\alpha(\mathfrak{P}_0)$ will be denoted the function α *reduced to be on* $\mathfrak{P}_0 = [p_0]$, a subclass of $\mathfrak{P}$; that is, α considered only for elements p_0 of $\mathfrak{P}_0$. Every reduced function $\alpha(\mathfrak{P}_0)$ gives rise to a reduced limit $L\alpha(\mathfrak{P}_0)$, if R as on $\mathfrak{P}_0\mathfrak{P}_0$ has the property C.* The definition of $L\alpha(\mathfrak{P}_0)$ is the same as that of $L\alpha$, except that all elements p must now be restricted to be of $\mathfrak{P}_0$.

By $\mathfrak{A}^*$ will be denoted $\mathfrak{A}$ enlarged by the addition of $+\infty$ and $-\infty$. Any element of $\mathfrak{A}^*$ will be denoted by a^*.

0. If $L_R\alpha = a^*$, then $L_{R_*}\alpha = a^*$, and conversely.

1. *If* $L\alpha = a_1^*$ *and* $L\alpha = a_2^*$, *then* $a_1^* = a_2^*$.

By proper choice of notation the possible cases may be reduced to six:

1) a_1^* and a_2^* both finite.

2σ) $a_1^* = \sigma\infty, \qquad a_2^* = \sigma\infty.$

3σ) $a_1^* = \sigma\infty, \qquad a_2^*$ finite.

4) $a_1^* = +\infty, \qquad a_2^* = -\infty.$

Of these 3σ and 4 lead to contradictions, leaving only (1) to be considered since 2σ are in harmony with the theorem. Let us consider (1).

By hypothesis there are systems $(p_{1e}|e)$, $(p_{2e}|e)$ such that

$$|\alpha(p) - a_1^*| \leqq \frac{e}{2} \qquad (pRp_{1e}),$$

$$|\alpha(p) - a_2^*| \leqq \frac{e}{2} \qquad (pRp_{2e}).$$

By R^C there then exists a system $(p_e|e)$ such that $(p_eRp_{1e},\ p_eRp_{2e})$ for every e. Then

$$|a_1^* - a_2^*| \leqq |a_1^* - \alpha(p_e)| + |\alpha(p_e) - a_2^*| \leqq e$$

for every e, so that $|a_1^* - a_2^*| = 0$ and $a_1^* = a_2^*$.

2. *If* $L\alpha = a$, *then* $L|\alpha| = |a|$.†

3. *If* $L\alpha = a$, *then* $L(c\alpha) = ca \quad (c)$.

4. *If* $L\alpha_1 = a_1$, $L\alpha_2 = a_2$, *then*

$$L(\alpha_1 + \alpha_2) = a_1 + a_2,$$

$$L(\alpha_1\alpha_2) = a_1a_2,$$

$$L\frac{\alpha_1}{\alpha_2} = \frac{a_1}{a_2} \qquad (a_2 \neq 0).$$

---

* The relation R as on $\mathfrak{P}_0\mathfrak{P}_0$ is necessarily T.

† Here $|\alpha|$, *the absolute of* α, denotes the function: $|\alpha|(p) = |\alpha(p)| \qquad (p)$.

5. *If* $\mathrm{L}\alpha$ *exists and, for certain* cdp_0, $|\alpha(p) - c| \leqq d$ *for* $p\mathrm{R}p_0$, *then* $|\mathrm{L}\alpha - c| \leqq d$.

6. *If* $\alpha(p) \geqq c$ *for every* p *and* $\mathrm{L}\alpha$ *exists, then* $\mathrm{L}\alpha \geqq c$.

The proofs of propositions 2–6 are simple.

A single-valued function α is *monotone increasing* (*relative to* R) if $\alpha(p_1) \geqq \alpha(p_2)$ for every pair (p_1, p_2) such that $p_1\mathrm{R}p_2$; *properly* so if $\alpha(p_1) > \alpha(p_2)$ for every pair of distinct elements (p_1, p_2) such that $p_1\mathrm{R}p_2$. The term *monotone decreasing* is similarly defined.

7. *If* α *is single-valued and monotone, then* $\mathrm{L}\alpha$ *exists finite or infinite and is equal to* $\overline{B}\alpha$* *or* $\underline{B}\alpha$, *that is,* $\overline{B}_p\alpha(p)$ *or* $\underline{B}_p\alpha(p)$, *according as* α *is monotone increasing or monotone decreasing.*

Assume first that α is monotone increasing and that $\overline{B}_p\alpha(p) = a$, a finite number. Then there exists a system $(p_e|e)$ such that

$$a \geqq \alpha(p_e) \geqq a - e \qquad (e).$$

Then

$$a \geqq \alpha(p) \geqq \alpha(p_e) \geqq a - e \qquad (p\mathrm{R}p_e) \qquad (e);$$

and hence,

$$|\alpha(p) - a| \leqq e \qquad (p\mathrm{R}p_e) \qquad (e);$$

so that $\mathrm{L}\alpha = a$. The other cases are treated similarly.

§ 2. Necessary and Sufficient Conditions for the Existence of a Limit.

1. (Cauchy Condition). *In order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that there shall exist a system* $(p_e|e)$ *such that*

$$|\alpha(p_1) - \alpha(p_2)| \leqq e \qquad (p_1\mathrm{R}p_e,\ p_2\mathrm{R}p_e).$$

This condition is *necessary*. For there exists a system $(p_e|e)$ such that

$$|\mathrm{L}\alpha - \alpha(p)| \leqq \frac{e}{2} \qquad (p\mathrm{R}p_e) \qquad (e).$$

The condition is also *sufficient*. For there exists a sequence $\{p_n\}$ such that $p_{n+1}\mathrm{R}p_n$ $(n = 1, 2, 3, \cdots)$ and

$$|\alpha(p') - \alpha(p'')| \leqq \frac{1}{2^n}$$

for $(p'\mathrm{R}p_n,\ p''\mathrm{R}p_n)$. The numbers $\alpha(p_1)$, $\alpha(p_2)$, $\cdots$ form a limited or bounded set since

$$|\alpha(p_n) - \alpha(p_2)| \leqq |\alpha(p_n) - \alpha(p_{n-1})| + \cdots + |\alpha(p_3) - \alpha(p_2)|$$
$$\leqq \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^1} \leqq 1,$$

for every $n > 2$. Hence there exists a subsequence $\{p_{n_m}\}$ such that the

* Read: *the (least) upper bound of* α.

numerical sequence $\{\alpha(p_{n_m})|m\}$ approaches some number a as a limit.* That $L\alpha = a$ now follows from the inequality

$$|\alpha(p) - a| \leqq |\alpha(p) - \alpha(p_{n_m})| + |\alpha(p_{n_m}) - a|$$
$$\leqq \frac{1}{2^{n_m}} + |\alpha(p_{n_m}) - a|,$$

which holds for $p\mathrm{R}p_{n_m}$.

The above theorem is true if R is replaced by R_* in the final line, as is evident by § 1, 0.

2. *In order that* $L\alpha$ *shall exist it is necessary and sufficient that*

$$L_{p_1p_2}[\alpha(p_1) - \alpha(p_2)] = 0,$$

that is, that there exist systems $(p_{1e}|e)$, $(p_{2e}|e)$ such that

$$|\alpha(p_1) - \alpha(p_2)| \leqq e \qquad (p_1\mathrm{R}p_{1e},\ p_2\mathrm{R}p_{2e}) \qquad (e).$$

This theorem follows at once from the preceding and the composition property of R.

3. *In order that* $L\alpha$ *shall exist it is necessary and sufficient that there exist a system* $(p_e|e)$ *such that*

$$|\alpha(p) - \alpha(p_e)| \leqq e \qquad (p\mathrm{R}p_e) \qquad (e).$$

The necessity of this condition follows from the R_* form of theorem 1 and its sufficiency also on taking the p_e of the required $(p_e|e)$ of that theorem as the given $p_{e/2}$ of the $(p_e|e)$ of the present theorem.

4. *In order that* $L\alpha$ *shall not exist it is necessary and sufficient that there exist* e_0 *and a system* $(p_{1p}, p_{2p}|p)$ *such that*

$$p_{1p}\mathrm{R}p,\ p_{2p}\mathrm{R}p,\ |\alpha(p_{1p}) - \alpha(p_{2p})| > e_0\dagger \qquad (p).$$

5. *In order that* $L\alpha$ *shall not exist it is necessary and sufficient that there exist* e_0 *and a system* $(p_p|p)$ *such that*

$$p_p\mathrm{R}p,\ |\alpha(p) - \alpha(p_p)| > e_0\dagger \qquad (p).$$

Theorems 4 and 5 follow from 1 and 3 respectively.

The following theorems involve sequences $\{p_n\}$ and a property: *monotone* (R), of sequences and two binary relations R_0, R on the class of sequences. A sequence $\{p_n\}$ is *monotone* (R) in case $p_{n+1}\mathrm{R}p_n$ (n). The notation $\{p'_n\}\mathrm{R}_0\{p''_n\}$ means that $p'_n\mathrm{R}p''_n$ (n). The notation $\{p'_n\}\mathrm{R}\{p''_n\}$ means that there exists a system $(n_n|n)$ such that $p'_{n_n}\mathrm{R}p''_n$ (n). Plainly if the relation R_0 holds for two sequences so does the relation R.

* The ordinary properties of $L_{n\to\infty}$ are here assumed known.

† If α is multiply valued, this inequality is understood to hold for at least one mode of determining the functional values involved.

6. *If* $\mathrm{L}\alpha = a^*$, *there exists a sequence* $\{p_n^0\}$ *monotone* (R) *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$, *and also for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}\mathrm{R}\{p_n^0\}$.

Take $\{p_n^0\}$ such that $p_{n+1}^0 \mathrm{R} p_n^0$ (n) and

$$|a^* - \alpha(p)| \leqq \frac{1}{n} \qquad (p\mathrm{R}p_n^0) \qquad (n)$$

or

$$\sigma\alpha(p) \geqq n \qquad (p\mathrm{R}p_n^0) \qquad (n),$$

according as a^* is finite or equals $\sigma\infty$, $\sigma = \pm$.

7. $\mathrm{L}\alpha = a^*$ *if there exists a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$.

We prove the equivalent contrapositive theorem:

7′. *If it is untrue that* $\mathrm{L}\alpha = a^*$, *then for every sequence* $\{p_n\}$ *there exists a sequence* $\{p_n^0\}$ *monotone* (R) *such that* $\{p_n^0\}\mathrm{R}_0\{p_n\}$ *and it is untrue that* $\mathrm{L}_n\alpha(p_n^0) = a^*$.

We have given e_0 and $(p_p | p)$ such that $p_p\mathrm{R}p$ (p) and (for at least one determination of $\alpha(p_p)$)

$$|a^* - \alpha(p_p)| > e_0 \qquad (p)$$

or

$$\sigma\alpha(p_p) < e_0 \qquad (p)$$

according as a^* is finite or equals $\sigma\infty$, $\sigma = \pm$. Hence from the given sequence $\{p_n\}$ an effective sequence $\{p_n^0\}$ is obtained by recursion in the form $\{p_{p'_n}\}$ on taking $p'_1 = p_1$ and for $n > 1$ $p'_n\mathrm{R}(p_n, p_{n-1}^0)$.

8. *The following conditions on* a^* *and the function* α *are equivalent*:

(*A*) $\mathrm{L}\alpha = a^*$;

(*B*) *There exists a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$;

(*C*) *There exists a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$;

(*D*) *There exists a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}\mathrm{R}\{p_n^0\}$.

This follows from 6 and 7. For by 6 A implies B and D; B implies C; D implies C; and by 7 C implies A.

9. *In order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that there exist a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n[\alpha(p_n^0) - \alpha(p_n)] = 0$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$.

The necessity follows from 8. The sufficiency is equivalent to

9′. *If* $\mathrm{L}\alpha$ *does not exist, then for every sequence* $\{p_n\}$ *there exists a sequence* $\{p_n^0\}$ *such that* $\{p_n^0\}\mathrm{R}_0\{p_n\}$ *and it is untrue that* $\mathrm{L}_n[\alpha(p_n) - \alpha(p_n^0)] = 0$.

In proof of 9′ we have given a sequence $\{p_n\}$ and by 5 a number e_0 and a system $(p_p|p)$ such that for every p $p_p\text{R}p$ and $|\alpha(p) - \alpha(p_p)| > e_0$.* Then, taking the sequence $\{p_{p_n}\}$ as sequence $\{p_n^0\}$, we have for every n $p_n^0\text{R}p_n$ and $|\alpha(p_n) - \alpha(p_n^0)| > e_0$,† and accordingly, as stated, $\{p_n^0\}\text{R}_0\{p_n\}$ and the untruth of $\text{L}_n[\alpha(p_n) - \alpha(p_n^0)] = 0$.

§ 3. Some Modes of Convergence.

The non-existence of $\text{L}\alpha$ implies by § 2, 5 the existence of a sequence‡ $\{p_n\}$ monotone (R) such that $\text{L}_n\alpha(p_n)$ does not converge, and for every sequence $\{p_n^0\}$ the existence of a sequence§ $\{p_n\}$ monotone (R) such that $\{p_n\}\text{R}\{p_n^0\}$ and $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| = \infty$. Hence we are led to formulate the following *modes of existence* (or *convergence*) of $\text{L}\alpha$.

$\text{L}\alpha$ *exists absolutely* in case there exists a sequence $\{p_n^0\}$ such that $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| < \infty$ for every sequence $\{p_n\}$ monotone (R) such that $\{p_n\}\text{R}\{p_n^0\}$.

$\text{L}\alpha$ *exists unconditionally* in case $\text{L}_n\alpha(p_n)$ converges finitely (or what is equivalent, in case $\Sigma_n[\alpha(p_{n+1}) - \alpha(p_n)]$ converges finitely) for every sequence $\{p_n\}$ monotone (R).

$\text{L}\alpha$ *exists absolutely-unconditionally* in case $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| < \infty$ for every sequence $\{p_n\}$ monotone (R).

$|\text{L}|\alpha = a$, *the absolute limit of* α *is* a, in case a is the least upper bound of $|\alpha(p_1)| + \Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)|$ for all sequences $\{p_n\}$ monotone (R). Here it would be simpler and equivalent to consider finite (instead of infinite) monotone sequences $\{p_n\}$.

By the initial remark, $\text{L}\alpha$ exists if $\text{L}\alpha$ exists absolutely or unconditionally. Evidently $\text{L}\alpha$ exists absolutely and unconditionally if $\text{L}\alpha$ exists absolutely-unconditionally. Finally, if $|\text{L}|\alpha$ exists, then $\text{L}\alpha$ exists absolutely-unconditionally and clearly $|\text{L}|\alpha \geqq |\text{L}\alpha|$.

That $\text{L}\alpha$ *exist unconditionally it is necessary and sufficient that* $\text{L}\alpha(\mathfrak{P}_0)$ *exist for every subclass* $\mathfrak{P}_0$ *of* $\mathfrak{P}$ *such that* R *as on* $\mathfrak{P}_0\mathfrak{P}_0$ *has the property* C. The condition is necessary, since $\text{L}\alpha$, and *a fortiori* $\text{L}\alpha(\mathfrak{P}_0)$, exists unconditionally, and therefore $\text{L}\alpha(\mathfrak{P}_0)$ exists. To prove it sufficient, that is, that $\text{L}_n\alpha(p_n)$ exists for every $\{p_n\}$ monotone (R), take $\mathfrak{P}_0 = [p_n|n]$. Then there exists a system $(n_e|e)$ such that $|\text{L}\alpha(\mathfrak{P}_0) - \alpha(p_n)| \leqq e$ for every $p_n\text{R}p_{n_e}$, in particular, for every p_n such that $n \geqq n_e + 1$. Hence $\text{L}_n\alpha(p_n)$ exists.

It is readily seen from the second paragraph of § 3 that if $\text{L}\alpha$ exists then as to the existence of $\text{L}\alpha$ absolutely, $\text{L}\alpha$ unconditionally, $\text{L}\alpha$ absolutely-

* For at least one determination of $\alpha(p)$ and $\alpha(p_p)$.

† For at least one determination of $\alpha(p_n)$, $\alpha(p_n^0)$.

‡ Take p_1 at random and for every n $p_{n+1} = p_{p_n}$.

§ Take p_1 at random, $p_2\text{R}(p_1, p_1^0, p_2^0)$, and for every n $p_{2n+1} = p_{p_{2n}}$, $p_{2n+2}\text{R}(p_{2n+1}, p_{2n+1}^0, p_{2n+2}^0)$.

unconditionally, $|L|\alpha$, only the six situations can occur which are indicated in the table below.

	$L\alpha$	$L\alpha$ abs.	$L\alpha$ unc.	$L\alpha$ abs.-unc.	$\lvert L\rvert\alpha$
(I)	+	+	+	+	+
(II)	+	+	+	+	−
(III)	+	+	+	−	−
(IV)	+	+	−	−	−
(V)	+	−	+	−	−
(VI)	+	−	−	−	−

Here a + sign indicates the existence of the concept at the head of the column, a − sign its non-existence. That these six situations actually occur is shown by the following examples (I) $\cdots$ (VI).

Let $A = a_1 + a_2 + \cdots = \mathrm{L}_n A_n$ be an absolutely convergent series. Let $C = c_1 + c_2 + \cdots = \mathrm{L}_n C_n$ be a conditionally convergent series. Denote by $\mathfrak{P}^{\mathrm{III}}$ the class $[n]$ of positive integers n.

(I) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case $n_1 \geqq n_2$. $\alpha(n) = A - A_n\ (n)$.

(II) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}} + \infty$. $n_1 \mathrm{R} n_2$ in case $n_1 = n_2$; $\infty \mathrm{R} \infty$; $\infty \mathrm{R} n\ (n)$. $\alpha(n) = n\ (n)$; $\alpha(\infty) = 0$.

(III) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case either n_1 odd n_2 even or $n_1 \equiv n_2 \pmod 2$ with $n_1 \geqq n_2$. $\alpha(n) = C - C_{(n+1)/2}$ (n odd), $A - A_{n/2}$ (n even).

(IV) The same as (III) except that $\alpha(n) = n$ (n odd), $A - A_{n/2}$ (n even).

(V) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case $n_1 \geqq n_2$. $\alpha(n) = C - C_n\ (n)$.

(VI) The same as (III) except that $\alpha(n) = n$ (n odd), $C - C_{n/2}$ (n even)

§4. Quasi-Limits. Upper and Lower Limits.

$L\alpha = a^*$, a quasi-limit of α is a^*, in case there exists a system $(p_{ep}|ep)$ such that $p_{ep}\mathrm{R}p\ (ep)$ and

$$|a^* - \alpha(p_{ep})| \leqq \epsilon \quad (ep) \qquad \text{or} \qquad \sigma\alpha(p_{ep}) \geqq e \quad (ep),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

$L_0\alpha = a^*$, a weak quasi-limit of α is a^*, in case there exists a system $(p_e|e)$ such that

$$|a^* - \alpha(p_e)| \leqq e \quad (e) \qquad \text{or} \qquad \sigma\alpha(p_e) \geqq e \quad (e),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

If $\mathfrak{P}_0$ is a subclass of $\mathfrak{P}$ such that $\mathfrak{P}$ as on $\mathfrak{P}_0\mathfrak{P}_0$ has the properties TC, the notations $L\alpha(\mathfrak{P}_0)$, $L_0\alpha(\mathfrak{P}_0)$ have obvious meanings. For two subclasses $\mathfrak{P}_1\mathfrak{P}_2$ of $\mathfrak{P}$ denote by $\mathfrak{P}_1\mathrm{R}\mathfrak{P}_2$ the condition that for every p_2 of $\mathfrak{P}_2$ there is a p_1 of $\mathfrak{P}_1$ for which $p_1\mathrm{R}p_2$. Hence if $\mathfrak{P}_0\mathrm{R}\mathfrak{P}$ R as on $\mathfrak{P}_0\mathfrak{P}_0$ has the properties TC.

1. *If $L\alpha = a^*$, then $L_0\alpha = a^*$.*

2. *In order that* $\mathrm{L}\alpha = a^*$, *it is necessary and sufficient* that $L\alpha(\mathfrak{P}_0) = a^*$ *uniquely for every* $\mathfrak{P}_0\mathrm{R}\mathfrak{P}$.

The condition is *necessary*. Since $L\alpha = a^*$ and $\mathfrak{P}_0 \mathrm{R} \mathfrak{P}$ there is a system $(p_e|e)_0$† such that

$$|a^* - \alpha(p)| \leqq e \quad (p\mathrm{R}p_e) \qquad \text{or} \qquad \sigma\alpha(p) \geqq e \quad (p\mathrm{R}p_e),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$. Then by means of a system $(p_{ep}|ep)_0$ such that $p_{ep}\mathrm{R}(p, p_e)$ $(ep)_0$, it is clear that $L\alpha(\mathfrak{P}_0) = a^*$. It remains to show that $L\alpha(\mathfrak{P}_0) = a_1^*$ implies $a_1^* = a^*$. From this hypothesis there is a system $(p_{1ep}|ep)_0$ such that we have for every e

$$|a_1^* - \alpha(p_{1ep_e})| \leqq \frac{e}{3} \qquad \text{or} \qquad \sigma_1\alpha(p_{1ep_e}) \geqq \frac{e}{3};$$

$$|a^* - \alpha(p_{1ep_e})| \leqq e \qquad \text{or} \qquad \sigma\alpha(p_{1ep_e}) \geqq e$$

(according to the values of a_1^*; a^*), whence the conclusion $a_1^* = a^*$ follows readily.

The condition is *sufficient*. The proof is indirect. The untruth of $L\alpha = a^*$ implies the existence of a positive number e_0 and a system $(p_p|p)$ such that

$$p_p\mathrm{R}p \quad (p), \qquad |\alpha(p_p) - a^*| > e_0 \quad (p) \qquad \text{or} \qquad \sigma\alpha(p_p) < e_0 \quad (p),$$

according to the value of a^*. Since $L\alpha = a^*$, $\mathfrak{P}$ being a $\mathfrak{P}_0\mathrm{R}\mathfrak{P}$, there is a system $(p_{ep}|ep)$ such that

$$p_{ep}\mathrm{R}p \quad (ep), \qquad |\alpha(p_{ep}) - a^*| \leqq e \quad (ep) \qquad \text{or} \qquad \sigma\alpha(p_{ep}) \geqq e \quad (ep),$$

according to the value of a^*. The class

$$\mathfrak{P}_0 \equiv [p_0] \equiv [p'_{ep} \equiv p_{p_{ep}}|ep]$$

is a class $\mathfrak{P}_0\mathrm{R}\mathfrak{P}$. Hence by hypothesis $L\alpha(\mathfrak{P}_0) = a^*$, whereas evidently

$$|\alpha(\mathfrak{P}_0) - a^*| > e_0 \qquad \text{or} \qquad \sigma\alpha(\mathfrak{P}_0) < e_0.$$

This is the desired contradiction. Hence $L\alpha = a^*$, as stated.

We assume α to be real- and single-valued in the remainder of § 4.

Associated with α are two functions $\overline{\alpha}$; $\underline{\alpha}$ (read: α *upper*; α *lower*) on $\mathfrak{P}$ to $\mathfrak{A}^*$, with

$$\overline{\alpha}(p) \equiv \overline{B}_{p_1|p_1\mathrm{R}p}\, \alpha(p_1); \qquad \underline{\alpha}(p) \equiv \underline{B}_{p_1|p_1\mathrm{R}p}\, \alpha(p_1),$$

that is, $\overline{\alpha}(p)$ is the (least) upper bound of $\alpha(p_1)$ for all $p_1\mathrm{R}p$; $\underline{\alpha}(p)$ is the (greatest) lower bound of $\alpha(p_1)$ for all $p_1\mathrm{R}p$. Plainly $\overline{\alpha}$; $\underline{\alpha}$ are monotone functions decreasing; increasing, and for every p $\underline{\alpha}(p) \leqq \overline{\alpha}(p)$.

The lower bound $\underline{B}\overline{\alpha} \equiv \underline{B}_p\overline{\alpha}(p)$ of the function $\overline{\alpha}$ is *the upper limit* $\overline{L}\alpha$ of the function α. Similarly the upper bound $\overline{B}\underline{\alpha}$ of $\underline{\alpha}$ is *the lower limit* $\underline{L}\alpha$ of α. It is readily seen that $\underline{L}\alpha \leqq \overline{L}\alpha$.

3. $L\overline{\alpha} = \overline{L}\alpha$; $L\underline{\alpha} = \underline{L}\alpha$.

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† In this proof the suffix 0 indicates that the elements p_e, p, p_{ep}, etc., involved belong to $\mathfrak{P}_0$.

This follows from § 1, 7.

4_1. *In order that* $\overline{\mathrm{L}}\alpha = a$, *it is necessary and sufficient that there exist systems* $(p_e|e)$, $(p_{ep}|ep)$ *such that*

(1) $$\alpha(p) \leqq a + e \quad (p\mathrm{R}p_e) \quad (e),$$

(2) $$p_{ep}\mathrm{R}p, \qquad \alpha(p_{ep}) \geqq a - e \quad (ep).$$

It is *necessary*. For there are systems $(p_e|e)$, $(p_{ep}|ep)$ such that $a \leqq \overline{\alpha}(p_e) \leqq a + e$ (e), $p_{ep}\mathrm{R}p$ (ep), $\overline{\alpha}(p) - \alpha(p_{ep}) \leqq e$ (ep). These systems satisfy (1), (2). For $\alpha(p) \leqq \overline{\alpha}(p_e) \leqq a + e$ $(p\mathrm{R}p_e)$ (e), and $a \leqq \overline{\alpha}(p) \leqq \alpha(p_{ep}) + e$ (ep).

It is *sufficient*. For by (1) $\overline{\alpha}(p_e) \leqq a + e$ (e); hence $\overline{\mathrm{L}}\alpha \leqq (\overline{\alpha}p_e) \leqq a + e$ (e), and accordingly $\overline{\mathrm{L}}\alpha \leqq a$; and by (2) $\overline{\alpha}(p) \geqq \alpha(p_{ep}) \geqq a - e$ (ep); hence $\overline{\alpha}(p) \geqq a$ (p), and accordingly $\overline{\mathrm{L}}\alpha \geqq a$. Hence $\overline{\mathrm{L}}\alpha = a$, as stated.

4_2. *In order that* $\underline{\mathrm{L}}\alpha = a$, *it is necessary and sufficient that there exist systems* $(p_e|e)$, $(p_{ep}|ep)$ *such that*

(1) $$\alpha(p) \geqq a - e \quad (p\mathrm{R}p_e) \quad (e),$$

(2) $$p_{ep}\mathrm{R}p, \qquad \alpha(p_{ep}) \leqq a + e \quad (ep).$$

5. $\mathrm{L}\alpha = a^*$ *is equivalent to* $\overline{\mathrm{L}}\alpha = \underline{\mathrm{L}}\alpha = a^*$.

6. $\boldsymbol{L}\alpha = \overline{\mathrm{L}}\alpha$, $\boldsymbol{L}\alpha = \underline{\mathrm{L}}\alpha$.

7. $\underline{\mathrm{L}}\alpha \leqq \boldsymbol{L}\alpha \leqq \overline{\mathrm{L}}\alpha$ *for every* $\boldsymbol{L}\alpha$ *finite or infinite.*

Theorems 5, 6, 7 follow readily* from the definitions and theorems 3, 4.

8. $\mathrm{L}\alpha = a^*$ *is equivalent to* $\boldsymbol{L}\alpha = a^*$ *uniquely.*

This theorem follows from 5, 6, 7.

§ 5. Limits as to Norm.

As basis for § 5 we take the system

$$\Sigma_1 = (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R};\ \nu),$$

where $\mathfrak{A}$, $\mathfrak{P}$, R have the same meanings as before and are subject to the same postulates, and where ν, *the norm*, is a real-valued function on $\mathfrak{P}$ to $\mathfrak{A}$ which is monotone decreasing (R) and such that $\underline{B}\nu = 0$.

$\mathrm{L}_\nu\alpha = a^*$, *the limit of* α *is* a^* *as to the norm* ν, provided there exists a system $(d_e|e)$ such that for every p such that $\nu(p) \leqq d_e$

$$|a^* - \alpha(p)| \leqq e \quad \text{or} \quad \sigma\alpha(p) \geqq e,$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

Associated with ν is a relation R' (having the properties T, C of the relation R) defined as follows: $p_1\mathrm{R}'p_2$ if and only if $\nu(p_1) \leqq \nu(p_2)$. It is

* E.g., if $\boldsymbol{\underline{L}}\alpha = a$, from the systems $(pe|e)(p_{ep}|_e p)$ of 4_2 we obtain a system $(p^\circ_{ep}|ep)$ effective for the proof of 6_2: $\boldsymbol{L}\alpha = \underline{\mathrm{L}}\alpha = a$, as follows: For every ep take $p'_{ep}\mathrm{R}(p,\ p_e)$ and $p^\circ_{ep} \equiv p_{ep'_{ep}}$.

easily shown that L_ν and $L_{R'}$ are equivalent: Whenever either $L_\nu\alpha$ or $L_{R'}\alpha$ exists finite or infinite, the other does and the two limits are equal. Hence the preceding theorems may be applied to L_ν. In particular L_ν is unique.

We are concerned in the sequel with certain relations between L_ν and L_R.

1. *If* $L_\nu\alpha = a^*$, *then* $L_R\alpha = a^*$.

Assume a^* finite. Then there exist systems $(d_e|e)$, $(p_e|e)$ such that

$$|a^* - \alpha(p)| \leqq e \qquad (p \text{ with } \nu(p) \leqq d_e) \quad (e)$$

and $\nu(p_e) \leqq d_e$ (e). Then $|a^* - \alpha(p)| \leqq e$ $(p\mathrm{R}p_e)$, as stated, since $\nu(p) \leqq \nu(p_e) \leqq d_e$ $(p\mathrm{R}p_e)$. The proof is similar for the case a^* infinite.

2. *If* $L_R\alpha$ *exists, then in order that* $L_\nu\alpha$ *shall exist and equal* $L_R\alpha$, *it is necessary and sufficient that there exist systems* $(d_e|e)$, $(p_{ep_1p}|ep_1p)$ *such that*

$$p_{ep_1p}\mathrm{R}p_1, \quad |\alpha(p_{ep_1p}) - \alpha(p)| \leqq e \quad (e,\ p_1,\ p \text{ with } \nu(p) \leqq d_e).$$

The condition is *necessary*. Take systems $(d_e|e)$, $(p_e|e)$, $(p_{ep_1p}|ep_1p)$ such that

$$|\alpha(p) - L_\nu\alpha| \leqq \frac{e}{2} \quad (p \text{ with } \nu(p) \leqq d_e) \quad (e),$$

$$\nu(p_e) \leqq d_e \quad (e),$$

$$p_{ep_1p}\mathrm{R}p_1, \qquad p_{ep_1p}\mathrm{R}p_e \quad (ep_1p).$$

(The p_{ep_1p} may clearly be taken as the same for every p.) Then for every p_1 and e and every p with $\nu(p) \leqq d_e$, we have

$$p_{ep_1p}\mathrm{R}p_1, \quad |\alpha(p) - \alpha(p_{ep_1p})| \leqq |\alpha(p) - L_\nu\alpha| + |L_\nu\alpha - \alpha(p_{ep_1p})| \leqq e,$$

as stated.

The condition is *sufficient*. Take $(p_e|e)$ such that $|\alpha(p) - L_R\alpha| \leqq e/2$ $(p\mathrm{R}p_e)$. Then for every e and p with $\nu(p) \leqq d_{e/2}$, we have

$$|\alpha(p) - L_R\alpha| \leqq |\alpha(p) - \alpha(p_{(e/2)p_ep})| + |\alpha(p_{(e/2)p_ep}) - L_R\alpha| \leqq e,$$

as stated.

All of the classical limits are limits as to a norm, and as such are instances of our general limit. We consider, as examples, the Riemann and Lebesgue integrals.

Let f denote a function of the variable x, x ranging over I: $a \leqq x \leqq b$. Take $\mathfrak{P}$ to be the class of all partitions p of I; a partition of I is a set $I_1, \cdots, I_n$ of intervals non-overlapping (except for end points) such that $I = I_1 + \cdots + I_n$. We define $\nu(p)$ for a partition $p = I_1, \cdots, I_n$ as the length of the longest I_k $(k = 1, \cdots, n)$. We say a partition $p' = I_1', \cdots, I_{n'}'$ is in the R relation to a partition $p'' = I_1'', \cdots, I_{n''}''$ if p' is a repartition of p'', that is, if every I_k' lies entirely in some I_k'' (except possibly for end points). Every function f gives rise to an associated (multiply

valued) function α on $\mathfrak{P}$ to $\mathfrak{A}$: $\alpha(p) \equiv \alpha(I_1, \cdots, I_n) \equiv \sum_k^{1n} f(x_k) I_k$, where I_k denotes the length (or measure) of I_k and x_k is any point of I_k. The ordinary Riemann definition of integration for a bounded function f is, except for form, as follows: If the function f is such that for the associated function α, $\mathrm{L}_\nu \alpha$ exists, then $\mathrm{L}_\nu \alpha$ is called the (Riemann) integral of f from a to b and is denoted by $\int_a^b f(x)dx$. But it follows from 1 and 2 above that $\mathrm{L}_{\mathrm{R}}\alpha$ used in place of $\mathrm{L}_\nu \alpha$ in the definition just given would yield a second and equivalent definition.

This remark is important in that it leads to a simple and natural definition of the Lebesgue integral $\int_a^b f(x)dx$ as a limit. To secure such a definition from the second definition it is only necessary to define partition differently, a partition now being a set of non-overlapping measurable sets $I_1, \cdots, I_n$ such that $I = I_1 + \cdots + I_n$. The junior author hopes soon to publish a theory of integration from this point of view which has been in his possession since 1917.

§ 6. Types of Uniform Convergence as to a General Parameter.

The *fundamental system* for § 6 is

$$\Sigma_2 \equiv (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\text{on } \mathfrak{P}\mathfrak{P}.TC};\ \mathfrak{Q}),$$

that is, the fundamental system of § 1 with the adjunction of a *general class* $\mathfrak{Q} \equiv [q]$ of elements q.

We consider functions $\alpha \equiv (\alpha(p) \,|\, p)$ on $\mathfrak{P}$ to $\mathfrak{A}$; $\beta \equiv (\beta(q) \,|\, q)$ on $\mathfrak{Q}$ to $\mathfrak{A}$; $\varphi \equiv (\varphi(pq) \,|\, pq)$ on $\mathfrak{P}\mathfrak{Q}$ to $\mathfrak{A}$. Thus a function φ is a function of the variables pq which range independently over the (conceptually, but not necessarily actually, distinct) classes $\mathfrak{P}\mathfrak{Q}$; we denote by $\varphi(\diamond q) \equiv (\varphi(pq) \,|\, p)$, $\varphi(p\diamond) \equiv (\varphi(pq) \,|\, q)$ the functions α on $\mathfrak{P}$ to $\mathfrak{A}$, β on $\mathfrak{Q}$ to $\mathfrak{A}$ obtained from φ by fixing the respective arguments q, p. With respect to the limits now to be defined the argument q plays the rôle of a parameter.

$\mathrm{L}\varphi = \beta$ ($\mathfrak{Q}$; unif.), *the limit of φ is β over $\mathfrak{Q}$ uniformly,* in case there exists a system $(p_e \,|\, e)$ such that

$$|\varphi(p\diamond) - \beta| \leqq e \quad (p\mathrm{R}p_e) \quad (e).$$

$\mathrm{L}\varphi = \beta$ ($\mathfrak{Q}$; quasi-unif.), *the limit of φ is β over $\mathfrak{Q}$ quasi-uniformly,* in case there exists a system $(p_{eq} \,|\, eq)$ such that

(1) the set $[p_{eq} \,|\, q]$ is finite (e),
(2) $|\varphi(pq) - \beta(q)| \leqq e \quad (p\mathrm{R}p_{eq}) \quad (eq)$.

$\mathrm{L}\varphi = \beta$ ($\mathfrak{Q}$; semi-unif.), *the limit of φ is β over $\mathfrak{Q}$ semi-uniformly,* in case there exists a system $(p_{eq} \,|\, eq)$ such that

(1) the set $[p_{eq} \,|\, q]$ is (finitely or infinitely) denumerable (e),
(2) $|\varphi(pq) - \beta(q)| \leqq e \quad (p\mathrm{R}p_{eq}) \quad (eq)$.

$\boldsymbol{L}\varphi = \beta\ (\mathfrak{Q};\ \text{unif.})$, *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *uniformly,* in case there exists a system $(p_{ep}\,|\,ep)$ such that

$$p_{ep}\mathrm{R}p, \quad |\varphi(p_{ep}\diamond) - \beta| \leqq e \quad (ep).$$

$\boldsymbol{L}\varphi = \beta\ (\mathfrak{Q};\ \text{quasi-unif.})$, *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *quasi-uniformly,* in case there exists a system $(p_{epq}\,|\,epq)$ such that

$$\begin{aligned} &(1)\ \text{the set } [p_{epq}\,|\,q] \text{ is finite} \quad (ep),\\ &(2)\ p_{epq}\mathrm{R}p, \quad |\varphi(p_{epq}q) - \beta(q)| \leqq e \quad (epq). \end{aligned}$$

$\boldsymbol{L}\varphi = \beta\ (\mathfrak{Q};\ \text{semi-unif.})$, *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *semi-uniformly,* in case there exists a system $(p_{epq}\,|\,epq)$ such that

$$\begin{aligned} &(1)\ \text{the set } [p_{epq}\,|\,q] \text{ is denumerable} \quad (ep),\\ &(2)\ p_{epq}\mathrm{R}p, \quad |\varphi(p_{epq}q) - \beta(q)| \leqq e \quad (epq). \end{aligned}$$

From the definitions of the three quasi-limits $\boldsymbol{L}$ we obtain definitions of the corresponding *weak quasi-limits* $\boldsymbol{L}_0$ by omitting the subscript p and the conditions involving R.

These types of uniform convergence as to a parameter are for use in §§ 7, 9. Obviously the quasi-uniform convergence L and the uniform convergence L are equivalent, and the quasi-uniform convergences L, $\boldsymbol{L}$, $\boldsymbol{L}_0$ imply the semi-uniform convergences L, $\boldsymbol{L}$, $\boldsymbol{L}_0$ respectively.

§ 7. Double Limits.

From two systems

$\Sigma' \equiv (\mathfrak{A};\ \mathfrak{P}' \equiv [p'];\ \mathrm{R}'^{\text{on }\mathfrak{P}'\mathfrak{P}'.TC})$, $\Sigma'' \equiv (\mathfrak{A};\ \mathfrak{P}'' \equiv [p''];\ \mathrm{R}''^{\text{on }\mathfrak{P}''\mathfrak{P}''.TC})$ of the type studied in §§ 1–4 we form the *composite system*

$$\Sigma \equiv (\mathfrak{A};\ \mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}'';\ \mathrm{R} \equiv \mathrm{R}'\mathrm{R}''^{\text{on }\mathfrak{P}\mathfrak{P}.TC})$$

of the same type. Here $\mathfrak{P}'$ $\mathfrak{P}''$ are two general classes conceptually (but not necessarily actually) distinct; $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ is the product or composite class $[p] \equiv [p'p'']$ of all composite elements $p \equiv p'p''$; and $\mathrm{R} \equiv \mathrm{R}'\mathrm{R}''$ is the composite relation R on $\mathfrak{P}\mathfrak{P}$: $p_1\mathrm{R}p_2 \equiv p_1'p_1''\mathrm{R}p_2'p_2''$ in case $p_1'\mathrm{R}'p_2'$ and $p_1''\mathrm{R}''p_2''$.

We denote limits, quasi-limits, weak quasi-limits as to R'; R''; R of functions α'; α''; α on $\mathfrak{P}'$; $\mathfrak{P}''$; $\mathfrak{P}$ to $\mathfrak{A}$ by L', $\boldsymbol{L}'$, $\boldsymbol{L}_0'$; L'', $\boldsymbol{L}''$, $\boldsymbol{L}_0''$; L, $\boldsymbol{L}$, $\boldsymbol{L}_0$, respectively. For a function α on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ $\mathrm{L}'\alpha$ is said to exist in case for every p'' $\mathrm{L}'\alpha(\diamond p'')$ exists, and in this case $\mathrm{L}'\alpha$ denotes the function $(\mathrm{L}'\alpha(\diamond p'')\,|\,p'')$ on $\mathfrak{P}''$ to $\mathfrak{A}$; etc.

The *iterated double limits* $\mathrm{L}'\mathrm{L}''$, $\mathrm{L}''\mathrm{L}'$ and *quasi-limits* $\boldsymbol{L}'\mathrm{L}''$, $\boldsymbol{L}_0'\mathrm{L}''$, $\boldsymbol{L}''\mathrm{L}'$, $\boldsymbol{L}_0''\mathrm{L}'$ have evident definitions. The *simultaneous double limit* $(\mathrm{L}'\mathrm{L}'') \equiv (\mathrm{L}''\mathrm{L}')$ and *quasi-limits* $(\boldsymbol{L}'\boldsymbol{L}'') \equiv (\boldsymbol{L}''\boldsymbol{L}')$, $(\boldsymbol{L}_0'\boldsymbol{L}_0'') \equiv (\boldsymbol{L}_0''\boldsymbol{L}_0')$ are defined as the simple limit L and quasi-limits $\boldsymbol{L}$, $\boldsymbol{L}_0$ for the composite

system Σ. The *simultaneous double limits* $(\boldsymbol{L}'\mathrm{L}'') \equiv (\mathrm{L}''\boldsymbol{L}')$, $(\boldsymbol{L}'_0\mathrm{L}'') \equiv (\mathrm{L}''\boldsymbol{L}'_0)$ are defined as follows:

$(\boldsymbol{L}'\mathrm{L}'')\alpha = a$, in case there exists a system $(p'_{ep'}, p''_{ep'} | ep')$ such that

$$p'_{ep'}\mathrm{R}'p', \quad |a - \alpha(p'_{ep'}p'')| \leqq e \quad (p''\mathrm{R}''p''_{ep'})(ep');$$

$(\boldsymbol{L}'_0\mathrm{L}'')\alpha = a$, in case there exists a system $(p'_e, p''_e | e)$ such that

$$|a - \alpha(p'_e p'')| \leqq e \quad (p''\mathrm{R}''p''_e) \quad (e).$$

The limits $(\boldsymbol{L}''\mathrm{L}')$, $(\boldsymbol{L}''_0\mathrm{L}')$ are defined similarly.

1. *If* $\mathrm{L}\alpha$ *and* $\mathrm{L}'\alpha$ *exist, then* $\mathrm{L}''\mathrm{L}'\alpha$ *exists and is equal to* $\mathrm{L}\alpha$.

2. *If* $\mathrm{L}'\alpha$ *and* $\mathrm{L}''\alpha$ *exist, then in order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that* $\mathrm{L}(\alpha - \mathrm{L}'\alpha) = 0$.

The condition is *necessary*.* For by 1 $\mathrm{L}''\mathrm{L}'\alpha$ exists, equal to $\mathrm{L}\alpha$. Hence $\mathrm{L}\mathrm{L}'\alpha$ exists, equal to $\mathrm{L}\alpha$, and

$$\mathrm{L}(\alpha - \mathrm{L}'\alpha) = \mathrm{L}\alpha - \mathrm{L}\mathrm{L}'\alpha = \mathrm{L}\alpha - \mathrm{L}\alpha = 0.$$

The condition is *sufficient*. From the hypotheses we readily secure a system $(p'_e, p''_{1e}, p''_{2e}, p''_e | e)$ such that for every e

$$|\alpha(p'p'') - \mathrm{L}'\alpha(\diamond p'')| \leqq \frac{e}{3} \quad (p'\mathrm{R}'_* p'_e, p''\mathrm{R}''_* p''_{1e}),$$

$$|\alpha(p'_e p''_1) - \alpha(p'_e p''_2)| \leqq \frac{e}{3} \quad (p''_1\mathrm{R}''p''_{2e}, p''_2\mathrm{R}''p''_{2e}),$$

$$p''_e\mathrm{R}''p''_{1e}, \quad p''_e\mathrm{R}''p''_{2e}.$$

Then for every e $(p'\mathrm{R}'p'_e, p''\mathrm{R}''p''_e)$ implies

$$\begin{aligned} |\alpha(p'p'') - \alpha(p'_e p''_e)| &\leqq |\alpha(p'p'') - \mathrm{L}'\alpha(\diamond p'')| \\ &\quad + |\mathrm{L}'\alpha(\diamond p'') - \alpha(p'_e p'')| + |\alpha(p'_e p'') - \alpha(p'_e p''_e)| \leqq e, \end{aligned}$$

and hence by § 2, 3 $\mathrm{L}\alpha$ exists, as stated.

3. *If* $\mathrm{L}'\alpha$ *exists* ($\mathfrak{P}''$; unif.) *and* $\mathrm{L}''\alpha$ *exists, then* $\mathrm{L}\alpha$, $\mathrm{L}'\mathrm{L}''\alpha$, $\mathrm{L}''\mathrm{L}'\alpha$ *exist and are equal.*

This theorem follows from 2 and 1, the uniformity implying the sufficient condition of 2.

4_1. *If* $\mathrm{L}''\alpha$ *exists, then in order that* $(\boldsymbol{L}'\mathrm{L}'')\alpha = a$ *it is necessary and sufficient that* $\boldsymbol{L}'\mathrm{L}''\alpha = a$.

It is *necessary*. We readily secure a system $(p'_{ep'}, p''_{ep'} | ep')$ such that

$$p'_{ep'}\mathrm{R}'p', \quad |a - \alpha(p'_{ep'}p'')| \leqq \frac{e}{2} \quad (p''\mathrm{R}''_* p''_{ep'}) \quad (ep'),$$

$$|\alpha(p'_{ep'}p''_{ep'}) - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq \frac{e}{2} \quad (ep').$$

* This proof of the necessity was suggested by Professor W. A. Hurwitz of Cornell University.

Then

$$|a - L''\alpha(p'_{ep'}\diamond)| \leqq |a - \alpha(p'_{ep'}p''_{ep'})| + |\alpha(p'_{ep'}p''_{ep'}) - L''\alpha(p'_{ep'}\diamond)| \leqq e \quad (ep').$$

It is *sufficient*. We secure a system $(p'_{ep'}, p''_{ep'} | ep')$ such that

$$p'_{ep'}R'p', \quad |a - L''\alpha(p'_{ep'}\diamond) \leqq \frac{e}{2} \qquad (ep'),$$

$$|L''\alpha(p'_{ep}\diamond) - \alpha(p'_{ep'}p'')| \leqq \frac{e}{2} \qquad (p''R''p''_{ep'}) \quad (ep').$$

Then

$$|a - \alpha(p'_{ep'}p'')| \leqq |a - L''\alpha(p'_{ep'}\diamond)| + |L''\alpha(p'_{ep'}\diamond) - \alpha(p'_{ep'}p'')| \leqq e \quad (p''R''p''_{ep'}) \quad (ep').$$

4_2. *If* $L'\alpha$ *exists, then in order that* $(L'\boldsymbol{L}'')\alpha = a$ *it is necessary and sufficient that* $\boldsymbol{L}''L'\alpha = a$.

5. *If* $L'\alpha$, $L''\alpha$, $L'L''\alpha$ *exist, then in order that* $L''L'\alpha$ *exist and equal* $L'L''\alpha$ *it is necessary and sufficient that* $(\boldsymbol{L}'L'')(\alpha - L'\alpha) = 0$.

It is *necessary*. Now $L''(\alpha - L'\alpha)$ exists equal to $L''\alpha - L''L'\alpha$, since each of thelatter limits exists. Since by § 4, 2 $\boldsymbol{L}'L''\alpha$ exists uniquely equal to $L'L''\alpha$ and obviously $\boldsymbol{L}'L''L'\alpha$ exists uniquely equal to $L''L'\alpha$, we have

$$\boldsymbol{L}'L''(\alpha - L'\alpha) = L'L''\alpha - L''L'\alpha = 0.$$

Hence by 4_1, $(\boldsymbol{L}'L'')(\alpha - L'\alpha) = 0$.

It is *sufficient*. We secure a system $(p'_{1e}, p'_{2e}, p''_e | e)$ such that for every e

$$|L'L''\alpha - L''\alpha(p'\diamond)| \leqq \frac{e}{3} \quad (p'R'p'_{1e}),$$

$$p'_{2e}R'p'_{1e}, \quad |\alpha(p'_{2e}p'') - L'\alpha(\diamond p'')| \leqq \frac{e}{3} \quad (p''R''_*p''_e).$$

Then

$$|L'L''\alpha - L'\alpha(\diamond p'')| \leqq |L'L''\alpha - L''\alpha(p'_{2e}\diamond)| + |L''\alpha(p'_{2e}\diamond) - \alpha(p'_{2e}p'')| + |\alpha(p'_{2e}p'') - L'\alpha(\diamond p'')| \leqq e \quad (p''R''p''_2) \quad (e).$$

6. *If* $L'\alpha$, $L''\alpha$ *exist, then in order that* $L'L''\alpha$ *shall exist it is sufficient that* $(L'\boldsymbol{L}''_0)(\alpha - L''\alpha) = 0$.

We secure a system $(p'_e, p''_e | e)$ such that for every e

$$|\alpha(p'p''_e) - L''\alpha(p'\diamond)| \leqq \frac{e}{3} \quad (p'R'_*p'_e),$$

$$|\alpha(p'p''_e) - \alpha(p'_ep''_e)| \leqq \frac{e}{3} \quad (p'R'p'_e).$$

Then

$$|L''\alpha(p'\diamond) - L''\alpha(p'_e\diamond)| \leqq |L''\alpha(p'\diamond) - \alpha(p'p''_e)| + |\alpha(p'p''_e) - \alpha(p'_ep''_e)| + |\alpha(p'_ep''_e) - L''\alpha(p'_e\diamond)| \leqq e \quad (p'R'p'_e) \quad (e).$$

7. *If* $L'\alpha$ *and* $L''\alpha$ *exist, then by* 5, 6 *the following four statements are mutually equivalent*:

$$\begin{aligned}&(1)\ L'L''\alpha \textit{ and } L''L'\alpha \textit{ exist and are equal};\\ &(2)\ L'L''\alpha \textit{ exists and } (\boldsymbol{L}'L'')(\alpha - L'\alpha) = 0;\\ &(3)\ L''L'\alpha \textit{ exists and } (\boldsymbol{L}''L')(\alpha - L''\alpha) = 0;\\ &(4)\ (\boldsymbol{L}'L'')(\alpha - L'\alpha) = 0,\ (L'\boldsymbol{L}_0'')(\alpha - L''\alpha) = 0.\end{aligned}$$

8. *If* $L'\alpha$, $L''\alpha$, $L'L''\alpha$ *exist, then in order that* $L''L'\alpha$ *shall exist and equal* $L'L''\alpha$ *it is sufficient that*

$$\boldsymbol{L}'(\alpha - L'\alpha) = 0 \quad (\mathfrak{P}'';\ \text{quasi-unif.}).$$

We secure systems $(p'_e|e)$, $(p'_{ep''}|ep'')$, $(p''_e|e)$ such that for every e

$$|L''\alpha(p'\diamond) - L'L''\alpha| \leqq \frac{e}{3} \quad (p'\mathrm{R}'p'_e),$$

$$[p'_{ep''}|p''] \text{ is finite},$$

$$p'_{ep''}\mathrm{R}'p'_e, \quad |L'\alpha(\diamond p'') - \alpha(p'_{ep''}p'')| \leqq \frac{e}{3} \quad (p''),$$

$$|\alpha(p'_{ep''}p'') - L''\alpha(p'_{ep''}\diamond)| \leqq \frac{e}{3} \quad (p''\mathrm{R}''p''_e).\dagger$$

Then for every e $p''\mathrm{R}''p''_e$ implies

$$\begin{aligned}|L'\alpha(\diamond p'') - L'L''\alpha| \leqq |L'\alpha(\diamond p'') - \alpha(p'_{ep''}p'')|\\ + |\alpha(p'_{ep''}p'') - L''\alpha(p'_{ep''}\diamond)| + |L''\alpha(p'_{ep''}\diamond) - L'L''\alpha| \leqq e,\end{aligned}$$

and accordingly by § 2, 3 $L''L'\alpha$ exists equal to $L'L''\alpha$, as stated.

§ 8. Lemmas: Revised Formulation of Certain Theorems of Fréchet.*

As foundation for § 8 we have the system

$$\Sigma_3 \equiv (\mathfrak{A};\ \mathfrak{Q};\ \mathfrak{S}^{1.2};\ L^{\text{on } \mathfrak{S} \text{ to } \mathfrak{Q}\cdot 1.2}),$$

where $\mathfrak{Q} \equiv [q]$ is a class of elements q; $\mathfrak{S} \equiv [s]$ is a class of sequences $s \equiv \{q_n\}$ of elements q_n of $\mathfrak{Q}$ to which belong 1) for every q of $\mathfrak{Q}$ the iterative sequence $\dot{q} \equiv \{q_n = q|n\}$ and 2) every subsequence s_0 of a sequence s of $\mathfrak{S}$; L is a single-valued function on $\mathfrak{S}$ to $\mathfrak{Q}$, associating with every sequence $s \equiv \{q_n\}$ of $\mathfrak{S}$ a definite element q, *the limit of* s, denoted by Ls, $L\{q_n\}$ or L_nq_n, for which 1) $L\dot{q} = q$ (q) and 2) $Ls_0 = Ls$ for every sequence s of $\mathfrak{S}$ and subsequence s_0 of s.

We denote by Q a non-null subset of the class $\mathfrak{Q}$. For every two sets Q_1 Q_2 the *sum* $Q_1 + Q_2$ and the *product* Q_1Q_2 are respectively the least

† This p''_e exists since $L''\alpha(p'\diamond)$ exists for every p' and the elements $p'_{ep''}$ for fixed e are finite in number.

* Fréchet, Sur quelques points du Calcul Fonctionnel, *Rendiconti . . . di Palermo*, Vol. 22 (1906).

common superset and the greatest common subset of the two sets, while the *difference* $Q_1 - Q_2$ is the set of all elements q of Q_1 but not of Q_2; thus $Q_1 + Q_2$ is a set Q while one (but not both) of the sets Q_1Q_2, $Q_1 - Q_2$ may be the null set.

A set Q is *closed* in case for every s (of $\mathfrak{S}$) in Q the limit Ls is of Q. A set Q is *compact* in case every infinite subset of Q contains an s (of $\mathfrak{S}$) consisting of distinct elements. A set Q_1 is *a region relative to a set* Q_2, in notation, a region (Q_2), in case every sequence s in Q_2 with Ls of Q_1 is ultimately in Q_1, i.e., every element of s after a certain one is an element of Q_1; accordingly, every set Q is a region (Q). A set Q is *covered by* a set [R] of regions (Q) in case every q of Q is of some R of [R]. A set Q is *enclosable* in case there exists a denumerable set [R] of regions (Q) such that 1) Q is covered by [R] and 2) if an element q of Q is an element of a region (Q), say Q_0, then there exists a region R of [R] contained in Q_0 and containing q.

1. *If* $\{Q_n\}$ *is a sequence of closed compact sets* Q_n, *each containing the following, then the product of all the sets* Q_n *contains at least one element* q.

2. *If a closed compact set* Q *is covered by a denumerable set* $[R_n | n]$ *of regions* (Q), *then the set* Q *is covered by some finite subset of the set* $[R_n | n]$.

For* otherwise, in contradiction to 1, $\{Q - (R_1 + \cdots + R_n)\}$ is a sequence $\{Q_n\}$ of closed compact sets, each containing the following, whose product is the null set, since every element q of Q is of some region R_n and hence not of the corresponding set Q_n.

3. *If an enclosable set* Q *is covered by a set* [R] *of regions* (Q), *then it is covered by some denumerable subset of the set* [R].

Denote by $[R_1]$ a denumerable set of regions (Q) covering the set Q in the sense of the enclosability of Q. An element q lies in a region R and accordingly in a region R_1 contained in R. Thus the set Q is covered by a necessarily denumerable subset $[R_2]$ of the set $[R_1]$ each of which is in a region R. Accordingly the set Q is covered by a denumerable subset $[R_3]$ of the set [R], as stated.

From 2, 3 we have the Heine-Borel-Lebesgue theorem:

4. *If a closed compact enclosable set* Q *is covered by a set* [R] *of regions* (Q), *then the set* Q *is covered by some finite subset of the set* [R].

A function β on Q to $\mathfrak{A}$ is *continuous at an element* q of Q, in case $L_n\beta(q_n) = \beta(q)$ for every sequence $\{q_n\}$ in Q with $L_nq_n = q$; and it is *continuous on* Q, in case it is continuous at every q of Q.

§ 9. Composite Range. Continuity.

The theorems of this section are with reference to the foundation.

$$\Sigma_4 = (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\text{on } \mathfrak{P}\mathfrak{P}\cdot TC};\ \mathfrak{Q};\ \mathfrak{S}^{1.2};\ \mathrm{L}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{Q}\cdot 1.2}),$$

* Compare Hausdorff, *Grundzüge der Mengenlehre*, p. 272.

where the notations have the same meanings as in §§ 1, 6, 8.

We consider a subset Q of $\mathfrak{Q}$ and a function φ on $\mathfrak{P}$Q to $\mathfrak{A}$.

1. *If the function* $\varphi(p\diamond)$ *is continuous on the set* Q *for every* p, *and* $\mathrm{L}\varphi(\diamond q)$ *exists for every* q *of* Q, *then in order that* $\mathrm{L}\varphi$ *shall be continuous on* Q *it is sufficient that*

$$\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0 \quad (\mathrm{Q};\ \text{quasi-unif.}).$$

Consider an element q of Q and a sequence $\{q_n\}$ of Q with $\mathrm{L}_n q_n = q$. We are to prove that $\mathrm{L}_n\mathrm{L}\varphi(\diamond q_n)$ exists and is equal to $\mathrm{L}\varphi(\diamond q)$. This is a consequence of § 7, 8. For $\mathrm{L}\varphi(\diamond q_n)$ exists for every n; $\mathrm{L}_n\varphi(pq_n)$ exists for every p; $\mathrm{LL}_n\varphi(\diamond q_n)$, quâ $\mathrm{L}\varphi(\diamond q)$, exists, and the sufficient condition stated implies

$$\boldsymbol{L}[\varphi(\diamond q_n) - \mathrm{L}\varphi(\diamond q_n)] = 0 \quad ([n];\ \text{quasi-unif.}).$$

2. *If the set* Q *is compact and closed, and the function* $\varphi(p\diamond)$ *is continuous on* Q *for every* p, $\mathrm{L}\varphi(\diamond q)$ *exists for every* q *of* Q, *and* $\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0$ (Q; semi-unif.), *then in order that* $\mathrm{L}\varphi$ *shall be continuous on* Q, *it is necessary that* $\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0$ (Q; quasi-unif.).

There is a set $(p_{1epq} | epq^{\text{ of Q}})$ such that

1) $[p_{1epq} | q^{\text{ of Q}}]$ is denumerable (ep),

2) $p_{1epq}\mathrm{R}p,\ |\varphi(p_{1epq}q) - \mathrm{L}\varphi(\diamond q)| \leqq \frac{e}{2} < e \quad (epq^{\text{ of Q}})$.

For every $(epq^{\text{ of Q}})$ let Q_{epq} denote the set of all elements q' of Q for which

$$|\varphi(p_{1epq}q') - \mathrm{L}\varphi(\diamond q')| < e.$$

Every set Q_{epq} is a region (Q). Further for every (ep) the set $[\mathrm{Q}_{epq} | q^{\text{ of Q}}]$ is a denumerable set of regions (Q) covering Q. Hence by § 8, 2 there is for every (ep) a finite number of them

$$[\mathrm{Q}_{epq_1}, \mathrm{Q}_{epq_2}, \cdots]$$

which cover Q; for every q of Q set $p_{epq} \equiv p_{1epq_i}$ where i is the smallest integer such that Q_{epq_i} contains q. Then the system $(p_{epq} | epq^{\text{ of Q}})$ is effective in proof of the desired quasi-uniformity.

3. *If the set* Q *is enclosable and the function* $\varphi(p\diamond)$ *is continuous on* Q *for every* p *and* $\mathrm{L}\varphi(\diamond q)$ *exists for every* q *of* Q, *then if* $\mathrm{L}\alpha$ *is continuous on* Q *it is true that*

$$\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0 \quad (\mathrm{Q};\ \text{semi-unif.}).$$

For since $\mathrm{L}(\varphi - \mathrm{L}\varphi) = \mathrm{L}\varphi - \mathrm{LL}\varphi = \mathrm{L}\varphi - \mathrm{L}\varphi = 0$ (Q), we have $\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0$ (Q); hence there exists a set $(p_{1epq} | epq^{\text{ of Q}})$ such that

$$p_{1epq}\mathrm{R}p, \quad |\varphi(p_{1epq}q) - \mathrm{L}\varphi(\diamond q)| \leqq \frac{e}{2} < e \ (epq^{\text{ of Q}}).$$

For every $(epq^{\text{ of Q}})$ let Q_{epq} denote the set of all elements q' of Q for which $|\varphi(p_{1epq}q') - L\varphi(\diamond q')| < e$. Then every Q_{epq} is a region (Q). Further for every (ep) $[Q_{epq} | q^{\text{of Q}}]$ is a class of regions (Q) covering Q. Then by § 8, 3 there is a denumerable subset $[Q_{epq_1}, Q_{epq_2}, \cdots]$ of $[Q_{epq} | q^{\text{ of Q}}]$ covering Q; for every q of Q set $p_{epq} \equiv p_{epq_i}$, where i is the smallest integer such that Q_{epq_i} contains q. Then the system $(p_{epq} | epq^{\text{ of Q}})$ is effective in proof of the desired semi-uniformity.

From 1, 2, 3 we have the theorem of Arzelà:*

4. *If $\varphi(p\diamond)$ is continuous on* Q *for every p and* $L\varphi(\diamond q)$ *exists for every q of* Q, *then in order that* $L\varphi$ *shall be continuous on* Q *it is sufficient that* $L(\varphi - L\varphi) = 0$ (Q; quasi-unif.). *If* Q *is compact, closed and enclosable, then that condition is necessary.*

5. *If $\varphi(\diamond q)$ is monotone decreasing for every q of* Q, *it is true that*

(1) *if $L\varphi$ exists* (Q), *then* $L\varphi$ *exists* (Q);

(2) *if $L\varphi$ exists* (Q; semi-unif.), *then* $L\varphi$ *exists* (Q; semi-unif.).

(3) *if $L\varphi$ exists* (Q; quasi-unif.), *then* $L\varphi$ *exists* (Q; unif.).

This is easily proved. From 4, 5_3 we have at once the theorem of Dini:

6. *If* Q *is closed, compact and enclosable, and the function $\varphi(p\diamond)$ is continuous on* Q *for every p and $\varphi(\diamond q)$ is monotone decreasing for every q of* Q, *and* $L\varphi = 0$, *then* $L\varphi = 0$ (Q; unif.).

THE UNIVERSITY OF CHICAGO,
THE UNIVERSITY OF THE PHILIPPINES,
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* Arzelà, "Sulle Serie di Funzioni," *Rend. di Bologna*, Ser. V, Vol. 8, 1899.

SUBSTITUTION GROUPS WHOSE CYCLES OF THE SAME ORDER CONTAIN A GIVEN NUMBER OF LETTERS.

BY G. A. MILLER.

Let G be any transitive or intransitive substitution group, and let $a_1a_2\cdots a_k$ be any one of the cycles found in one or more of the substitutions of G. If this cycle and all of its conjugates under G are counted once for every substitution of G in which they appear, then the total number of letters in all of these cycles is exactly the order of G, since such a cycle is transformed into itself only by its powers by means of substitutions restricted to its own letters.* Hence the total number of letters in all the cycles of order k contained in G may be found as follows: Select a set of cycles of order k composed of all the different cycles of this order found in G. Let λ be the number of complete sets of conjugates under G contained in this set. The total number of letters in all the cycles of order k found in G is then λg, g being the order of G.

In particular, the holomorph of the cyclic group of order p, p being any prime number, contains $p-1$ cycles of order p. Since these cycles form a single set of conjugates under this holomorph $\lambda = 1$ in this case, and hence the order of this holomorph is $p(p-1)$, as is also otherwise evident. As another very elementary illustration it may be noted that every intransitive group of order p^m which has only transitive constituents of degree p involves only invariant cycles of order p. The number of the distinct cycles is $p-1$ times n/p, n being the degree of the group. Hence $\lambda = n(p-1)/p$ in this case and the total number of letters in all the substitutions of G is $n(p-1)p^{m-1}$ as is also evident from the fact that the average number of letters in all the substitutions of an intransitive group of degree n which has k transitive constituents is $n-k$.

A necessary and sufficient condition that a group G of degree n which has k systems of intransitivity has the property that the total number of letters in all of its cycles of the same order is exactly g for every cycle which appears in these substitutions is that G contains cycles of each of the orders $2, 3, \cdots, n-k+1$. This is clearly impossible when G is intransitive since the degree of each of these transitive constituents would be less than $n-k+1$. Moreover, it is clear that the symmetric group of degree n involves cycles of each of the orders $2, 3, \cdots, n$, and hence this group has the property that the total number of letters in all the cycles of the same

* E. Netto, *Journal für die reine und angewandte Mathematik*, Vol. 103 (1888), p. 323.

order is $n!$ for every cycle which appears in the group. It is easy to verify that the well-known triply transitive group of degree 6 and of order 120 has the same property.

To prove that this is the only non-symmetric group which has this property it may first be noted that if the degree of such a group would exceed 7 the group would involve a cycle of prime order p, where p satisfies the condition $n/2 < p \leqq n - 3$, according to a well-known theorem due to Tshebychef. Hence such a group must be primitive, and it cannot involve a cyclical substitution of order p without being either alternating or symmetric.* As the group could not be alternating since in the alternating group of degree n there cannot be cycles of both the orders n and $n - 1$, it must be symmetric.

It remains therefore only to prove that there is no transitive group of degree n less than 8, besides the triply transitive group of degree 6 to which we have already referred, which has the property that its substitutions involve cycles of each of the orders 2, 3, $\cdots$, n. As these groups are well known this proof may be regarded as obvious, since it implies at most an examination of a few lists of substitution groups of low degrees.

To the many interesting known properties of the triply transitive group of degree 6, which was considered at length by A. L. Cauchy in volume 22 (1846) of the *Comptes Rendus*, and had been studied earlier by C. Hermite, we have here added one which all symmetric groups possess but which no other non-symmetric group possesses. This property therefore belongs to this group both when it is represented on 5 letters and when it is represented on 6 letters. In all other cases the symmetric group of degree n loses this property when it is represented on more than n letters.

The alternating group of degree n involves cycles of each of the orders 2, 3, $\cdots$, n except of order n, when n is even, or of order $n - 1$, when n is odd. Hence it results directly that the total number of letters in all the cycles of the same order contained in the alternating group is equal to the order of the group with the exception that this total is equal to twice the order of the group in the case of the largest cycle found in such a group. This follows directly from the fact that these largest cycles constitute two complete sets of conjugates under the alternating group. A necessary and sufficient condition that the holomorph of a cyclic group, written as a regular group, is composed of positive substitutions is that the orders of all the Sylow subgroups of this cyclic group are even powers of odd prime numbers.* This is evidently also a necessary and sufficient condition that

* G. A. Miller, *Bulletin of the American Mathematical Society*, Vol. 4 (1898), p. 141.

* This fundamental theorem relating to the holomorph of a cyclic group is not found in my article on this subject, *Transactions of the American Mathematical Society*, Vol. 4 (1903), p. 153.

all the generators of a cyclic group appear in one of the two complete sets of conjugates noted above.

Besides the alternating groups there are various others which have the property that the total number of letters in all the cycles of the same order is equal to the order of the group with the exception that for one such order this number is twice the order of the group. If such a group is transitive it must involve cycles of each of the orders 2, 3, $\cdots$, n except for one such order. This condition is evidently sufficient as well as necessary. It is easily seen that the degree of such a transitive group could not exceed 15 since there are at least two distinct prime numbers which satisfy the condition $n/2 < p \leqq n-3$ whenever n is a positive integer greater than 15. This fact can easily be verified by means of tables of primes extending to one million, and for larger values of n it results directly from known formulas. Cf. E. Landau, *Primzahlen,* Vol. 1 (1909), p. 91. In fact, from such formulas it results also that at least two such primes always exist when n exceeds a much smaller number than one million.

From the existence of at least two such primes it results directly, just as in the case of symmetric groups, that if the degree of a transitive group exceeds 15 it cannot have cycles of each of the orders 2, 3, $\cdots$, n, except one, without being alternating. It is easy to verify that no non-alternating group whose degree exceeds 8 satisfies these conditions and that the group of isomorphisms of the simple group of order 168 is the only transitive group of degree 8 which is not alternating but satisfies the conditions in question. In this group of order 336, just as in the alternating groups, the number of letters in all of the cycles of the same order, except the largest one, is equal to the order of the group, while the number of letters in the cycles of order 8 is equal to 672.

There is no transitive group of degree 7 which satisfies the conditions in question but there are two such groups of degree 6, viz., the largest groups which contain 2 or 3 systems of imprimitivity. These groups are of orders 72 and 48 respectively, and the number of letters in all their cycles of lowest order is twice the order of the group while the number of letters in all of their cycles of each other order is equal to the order of the group. On 5 letters there is clearly only one such non-alternating transitive group, viz., the group of order 20, while on 4 letters the cyclic group and the octic group constitute the only instances. In the cyclic group and the group of order 20 the number of letters in the cycles of order four is equal to twice the order of the group while in the octic group this is the case as regards the cycles of lowest order.

Hence there are just six transitive groups which are non-alternating but have the property that the total number of letters in all the cycles of each

order save one is the order of the group while for this one the total number of letters is twice the order of the group. Three of these have in common with all the alternating groups the property that the cycles in which the largest number of letters appear are also the largest cycles, while in the other three the largest number of letters appear in the smallest cycles.

If in a constituent of an intransitive group the total number of letters found in all of its cycles of the same order is k times the order of this constituent, then the total number of letters found in these cycles for all the substitutions of the entire group is also equal to k times the order of this group. In particular, if each of k constituents of an intransitive group contains a cycle of the same order, then the total number of letters in all the cycles of this order contained in the group is at least k times the order of the group. Hence it results that if an intransitive group has the property that the total number of letters found in its cycles of the same order, except one, is equal to the order of the group and for this one it is equal to twice this order, then no more than two of its transitive constituents can involve cycles of the same order. If two such constituents involve cycles of the same order, the total number of letters in all the cycles of the same order found in these constituents must be the order of the constituent.

It is evident that the intransitive groups which have for one constituent the regular group of order 2 and for the other any symmetric group, or the triply transitive group of degree 6, satisfy the condition that the total number of letters found in their cycles of order 2 is twice the order of the group while the total number of letters in all of the other cycles of the same order is equal to the order of the group. Moreover, these are the only intransitive groups in which at least two transitive constituents have cycles of the same order and which, moreover, satisfy the condition that the total number of letters found in the cycles of one order is twice the order of the group while the total number of letters found in all the cycles of every other order is equal to the order of the group. If no two transitive constituents have cycles of the same order, the orders of these constituents must be relatively prime.

As the orders of all the transitive groups which have the property that either the total number of letters in the cycles of the same order is equal to the order of the group for every order of such a cycle, or this is true of all cycles except those of one order in which the total number of letters involved is twice the order of the group, are even with the exception of the alternating group of degree 3, it remains therefore only to consider the intransitive groups having for one constituent this alternating group. The other constituent is evidently the symmetric group of order 2, and hence the intransitive cyclic group of degree 5 and order 6 is the only intransitive

group which has the property that the total number of letters in every cycle of the same order except one is equal to the order of the group and that the number of letters in the cycles of this particular order is equal to twice the order of the group while the orders of the transitive constituents are relatively prime.

If no two of the complete set of conjugates of a given cycle of a transitive group have a common letter, then these conjugates constitute a substitution which is invariant under the transitive group. Moreover, if a substitution is invariant under a transitive group, its subgroup which is composed of all its substitutions omitting one letter must omit more than one letter, and hence the group must *involve* a cycle whose conjugates under the group have no common letter. That is, *a necessary and sufficient condition that there is a substitution, besides the identity, which is commutative with every substitution of a transitive group is that the group involves a cycle such that no two of the conjugates of this cycle under the group have a common letter.* The total number of substitutions which are commutative with every substitution of a transitive group is therefore equal to the number of its different cycles which involve the same letter and have the property that no two of the cycles in any one of the complete sets of conjugates to which they separately belong have a common letter.

As the total number of letters found in identical cycles and all their conjugates is equal to the order of the group, it results directly that the number of cycles which are identical with a cycle having the property that no two of its conjugates involve the same letter is equal to the order of the group divided by its degree. In particular, the number of such cycles which are identical is an invariant of the substitution group. A necessary and sufficient condition that a transitive group is regular is that every one of its cycles has the property that no two of its conjugates under the group have a common letter. The $g - 1$ complete sets of conjugates of the cycles of a regular group G of order g give rise to $g - 1$ substitutions which are separately commutative with every substitution of G. The fact that these $g - 1$ substitutions together with the identity constitute a group follows directly from the facts that all the substitutions which are commutative with every substitution of G must constitute a group and that each such substitution besides the identity must be of degree g. We thus have a new proof of the fact that all the substitutions which are commutative with each substitution of a regular group of order g constitute a regular group of this order.

Since every transitive group whose order is of the form p^m, p being a prime number, involves invariant substitutions, it results that it involves cycles such that no two of their conjugates under the group involve a

common letter. When the group is non-regular each such cycle must appear in more than one substitution. In fact, the number of substitutions in which it appears is always equal to the order of the group divided by its degree even when the order of the group is not a power of a prime. In particular, when the degree of a group of order p^m is p^2, $m > 2$, its only cycles which appear in more than one substitution are those which are found in its invariant subgroup of order p. That is, *in every transitive group of degree p^2 and of order p^m, $m > 2$, the total number of letters found in the conjugates of every cycle which does not appear in the invariant subgroup of order p is equal to the order of the group.*

If a transitive group of degree n is regular or of class $n - 1$, it evidently cannot involve two substitutions which contain the same cycle. It is not difficult to prove that every other transitive group contains such substitutions. To prove this theorem, let G be any transitive group of degree n which involves at least one substitution s whose degree does not exceed $n - 2$. If G is at least doubly transitive, it must involve a cycle of order 2 which does not involve any of the letters of s. The subgroup generated by s and a substitution involving this cycle will clearly have this cycle in one half of its substitutions, and as the order of this subgroup exceeds 2 it may be assumed in what follows that G is simply transitive. It will also be convenient to assume that G is a group of lowest possible degree which does not involve two substitutions containing the same cycle but involves at least one substitution whose degree does not exceed $n - 2$.

The subgroup G_1 composed of all the substitutions of G which omit a given letter is of degree $n - \alpha$. If $\alpha > 1$, it is well known that G_1 is invariant under an intransitive group which has one transitive constituent of degree α. This intransitive group evidently involves at least two substitutions containing the same cycle. Hence it may be assumed that $\alpha = 1$ and that G_1 is formed by a simple isomorphism between transitive groups of which at least one is of a class one less than its degree while the others, if any, are regular. If a transitive group of degree k and of class $k - 1$ can be represented in more than one way as a transitive group whose degree is exactly one unit larger than its class, every invariant regular subgroup must include all the subgroups which appear in the regular form when the group is represented on a smaller number of letters. This follows directly from the fact that in a transitive group of degree k and of class $k - 1$ every substitution of degree $k - 1$ is transformed into k distinct substitutions by the substitutions of the regular subgroup of order k and this subgroup of order k must be found in every invariant subgroup which involves a substitution of degree $k - 1$.

The substitution s may be supposed to be found in G_1. If the group generated by s is transformed into itself by a substitution of G which is not

found in G_1, this substitution and s will clearly generate an intransitive group having a constituent whose order is less than the order of this intransitive group and therefore a cycle which appears in more than one of its substitutions. Hence it may be supposed that s is transformed under G into a substitution of G_1 with which it is not conjugate under G_1. As this substitution may be supposed to involve the same letters as s does, it results that the transforming substitution and s again generate an intransitive group involving a constituent whose order is less than the order of this intransitive group. Hence it has been proved that *a necessary and sufficient condition that a transitive group of degree n contains a complete set of distinct conjugate cycles whose total number of letters is less than the order of the group is that the class of the transitive group is less than* $n - 1$.

If a substitution group of degree n involves cycles of order n or of order $n - 1$, the total number of letters in a complete set of conjugates of such a cycle is always equal to the order of the group since such a cycle cannot appear in two different substitutions. This is a special case of the evident theorem that if a group of degree n involves a cycle of order $n - \alpha$ but no substitution on α or less than α letters, then the total number of letters in all the conjugates of this cycle must equal the order of the group. In particular, if a primitive substitution group of degree n involves a complete set of conjugate cycles of order $n - 2$ such that the total number of letters in all the cycles of the set is less than the order of the group, the group must be the symmetric group of degree n, and if an imprimitive group of degree n involves such a complete set of conjugates, n must be even and the group must involve the abelian subgroup which has for its independent generators $n/2$ transpositions.

In the preceding paragraph there appears a new definition of the symmetric group. As another such definition we give the following: The symmetric group of degree n is the transitive group of this degree in which each cycle of order 2 appears in $(n - 2)!$ substitutions. It may also be noted that if a multiply transitive group of degree n is not of class $n - 1$, then each of its cycles of order 2 must be found in more than one substitution. In fact, each such cycle can be transformed into every other such cycle since the group is at least two times transitive. Some such cycle must appear in more than one substitution, since the cycle of order 2 which involves the letter a, for instance, must involve another letter which does not appear in at least two of the substitutions of the subgroup composed of all the substitutions which omit a. Hence the theorem: *Every cycle of order 2 found in a multiply transitive group appears in more than one substitution whenever the class of this group is not one less than its degree.*

UNIVERSITY OF ILLINOIS

BOUNDARY VALUE AND EXPANSION PROBLEMS: OSCILLATION, COMPARISON AND EXPANSION THEOREMS.*

By R. D. Carmichael.

1. *Algebraic Oscillation and Comparison Theorems.*—On any convenient horizontal straight line segment, say the points s such that $a \leqq s \leqq b$, let us erect n perpendiculars two of which are at the ends of the segment while the other $n - 2$ are evenly or unevenly distributed on the interior of the segment. Let these be marked from left to right by the numbers 1, 2, $\cdots$, n; and consider them as analogous to the n coördinate axes of a space of n dimensions. Let the greatest distance between two consecutive axes be called the norm of the system of axes. Having given the set of real constants $u_1, u_2, \cdots, u_n$, let us take a point on the ith axis at a distance $|u_i|$ from the original segment and above it or below it according as u_i is positive or negative. Having done this for each value i of the set 1, 2, $\cdots$, n, join by straight line segments the point on each interior axis to the points on the two adjacent axes. We thus obtain a broken line which we shall call the graphic representation of the point $(u_1, u_2, \cdots, u_n)$ in space of n dimensions or of the set of constants $u_1, u_2, \cdots, u_n$. This broken line is the graph of a continuous function $u(s)$ of the real variable s on the interval $a \leqq s \leqq b$. We shall say that this function $u(s)$ is obtained from the set of constants $u_1, u_2, \cdots, u_n$ by *linear interpolation* with respect to the given n axes. The zeros of $u(s)$ we shall call the *zeros* of the set of constants with respect to the given system of coördinates.

Let us consider the system of n equations

$$(1.1) \qquad \sum_{j=1}^{n+2} a_{ij}x_j = 0, \qquad i = 1, 2, \cdots, n,$$

in the $n + 2$ unknown quantities $x_1, x_2, \cdots, x_{n+2}$, the matrix of coefficients of this system being of rank n. Let D_i denote the determinant of the matrix obtained from the matrix of coefficients in (1.1) by striking out the ith and $(i + 1)$th columns. We then have the following fundamental theorem:†

Theorem I. *Let D_i for a given range R of consecutive values of the integer i be of one sign and let I denote the interval of the s-axis corresponding to this range of i in the sense of the first paragraph above. Let u_i and v_i be any two linearly independent solutions of the system* (1.1) *the matrix of whose coefficients is of rank n; and let these solutions be extended, by the method of*

* Presented to the American Mathematical Society.

† *American Journal of Mathematics*, 43 (1921): 69–101; see p. 84.

linear interpolation employed above, to the functions $u(s)$ and $v(s)$. Then on the interval I the zeros of $u(s)$ and $v(s)$ separate each other; that is, between any two consecutive zeros (on I) of one of these functions there is one and just one zero of the other function.

On examining the proof of this theorem, in the article cited, it is seen that the only use made of the hypothesis on D_i is in showing that ω_i,

$$\omega_i = \begin{vmatrix} u_i & v_i \\ u_{i+1} & v_{i+1} \end{vmatrix},$$

is of one sign in I, so that the theorem might be restated with D_i replaced by ω_i in the first line. The theorem as first stated is in the more useful form for suggesting analogous theorems in the transcendental cases; but the second form will be found more suggestive for proofs.

The foregoing theorem is analogous to the Sturmian theorem of oscillation for homogeneous linear differential equations of the second order and indeed reduces to that theorem by a certain limiting process. The object of this paper is (*a*) to derive (§ 1) the algebraic theorems which are analogous to the Sturmian theorems of comparison for homogeneous linear differential equations of the second order and of which the latter are limiting forms, (*b*) to obtain (§ 2) theorems of oscillation for differential equations of order n and for certain functional equations including difference and q-difference equations, (*c*) to derive (§ 3) corresponding theorems of comparisons by aid of the named algebraic theorems of comparison, (*d*) to point out (§ 4) a certain generalization of boundary conditions for expansion problems, and (*e*) to indicate (§ 5) the character of certain expansion problems for q-difference and integro-q-difference equations.

In what follows in this section we shall assume that the notation in (1.1) is so chosen that the determinant D_1 is different from zero. If the equations are taken in a suitable order (and we shall suppose them so written already), it is obviously possible to combine them into a new system having the same solutions x in such way that the new coefficients a_{ij} have the value zero when $j > i + 2$ and such that every $a_{i,i+2}$ is different from zero; and we reduce the latter to unity by dividing both members of the ith equation by $a_{i,i+2}$. We shall now suppose further that the original, and hence the new, matrix $||a_{ij}||$ has the property that the determinants of orders $1, 2, \cdots, n$ in its upper left-hand corner are all different from zero in value. Then it is possible to make further combinations of equations in the system so as to arrive at a new system with the same solutions x and of such sort that the new coefficients a_{ij} are zero when $j < i$. Then, changing the order of terms in the equation, we have a system in the form

$$(1.2) \qquad x_{i+2} + \alpha_i x_{i+1} + \beta_i x_i = 0, \qquad i = 1, 2, \cdots, n,$$

where α_i and β_i are determinate functions of the original coefficients a_{ij}.

On writing $x_i = y_i u_i$, the foregoing equation reduces to the following:

$$(1.3) \qquad y_{i+2}u_{i+2} + \alpha_i y_{i+1}u_{i+1} + \beta_i y_i u_i = 0.$$

We set $y_{i+2} = \beta_i y_i$. If y_1 and y_2 are any two numbers different from zero, we have

$$y_{2i+1} = \beta_{2i-1}\beta_{2i-3}\cdots\beta_3\beta_1 y_1, \qquad y_{2i} = \beta_{2i-2}\beta_{2i-4}\cdots\beta_4\beta_2 y_2.$$

If β_i is positive for all i and if y_1 and y_2 are positive, it is clear that y_i is always positive and hence that u_i and x_i have always the same sign. Therefore the functions $x(s)$ and $u(s)$, obtained by linear interpolation from x_i and u_i with respect to a given system of coördinates, have their zeros on the same intervals of the coördinate system (though not necessarily at the same points). Equation (1.3) may be written in the form

$$(1.4) \qquad u_{i+2} + \varphi_i u_{i+1} + u_i = 0, \qquad \varphi_i = \alpha_i \frac{y_{i+1}}{y_{i+2}},$$

where φ_i is a determinate function of the a_{ij} in (1.1). From the principal properties of the distribution of the zeros of u (s) one knows the principal properties of the distribution of the zeros of x (s) provided that β_i is positive for each value of i.

A second equation of the form (1.1), under such hypotheses as we have just employed, would reduce to the normal form

$$(1.5) \qquad v_{i+2} + \psi_i v_{i+1} + v_i = 0.$$

Comparison theorems for the distribution of the zeros of the functions $u(s)$ and $v(s)$, obtained from the constants u_i and v_i by linear interpolation, yield corresponding theorems for the two original systems of form (1.1). We shall derive and state the results only for the normal forms (1.4) and (1.5).

Multiplying (1.4) member by member by v_{i+1} and (1.5) by $-u_{i+1}$ and adding, we have a result which may be put in the form

$$\Delta\{u_{i+1}v_i - u_i v_{i+1}\} + (\varphi_i - \psi_i)u_{i+1}v_{i+1} = 0;$$

whence, on summing as to i from μ to m, it follows that

$$(1.6) \quad (u_{m+2}v_{m+1} - u_{m+1}v_{m+2}) - (u_{\mu+1}v_\mu - u_\mu v_{\mu+1}) + \sum_{i=\mu}^{m} (\varphi_i - \psi_i)u_{i+1}v_{i+1} = 0,$$

where μ and m are any two numbers of the set $1, 2, \cdots, n$ such that $\mu \leqq m$.

If solutions u_i and v_i of (1.4) and (1.5), respectively, interpolate into functions $u(s)$ and $v(s)$ having a common zero, say on the μth interval, and if $u(s)$ has a zero on an interval to the right of the μth, say on the

$(m+1)$th interval, m being chosen as small as possible, we may show that $v(s)$ has a zero between these two zeros of $u(s)$ provided that $\varphi_i \leqq \psi_i$, $i = \mu, \mu+1, \cdots, m$, the equality sign not holding for all these values. Without loss of generality in argument we take $u(s)$ and $v(s)$ to be positive each on the interval from its zero on the μth interval to its next zero to the right. We shall prove the statement in consideration by showing that we are led to a contradiction if we suppose that $v(s)$ is nowhere negative between the two consecutive zeros of $u(s)$. We have $u_{\mu+1}v_\mu - u_\mu v_{\mu+1} = 0$, since $u(s)$ and $v(s)$ have a common zero on the μth interval. Then from (1.6) we see that we now have

$$u_{m+2}v_{m+1} - u_{m+1}v_{m+2} = \sum_{i=\mu}^{m} (\psi_i - \varphi_i)u_{i+1}v_{i+1} > 0.$$

This requires that $v(s)$ shall have a zero on the $(m+1)$th interval and to the left of that of $u(s)$, so that $v(s)$ is negative near the right-hand end of the interval between the named consecutive zeros of $u(s)$, contrary to hypothesis. Hence $v(s)$ has a zero between these two consecutive zeros of $u(s)$.

If we take any solution $\bar{v}_i$ of (1.5) linearly independent of v_i, then the corresponding function $\bar{v}(s)$ has a zero between two consecutive zeros of $v(s)$ by theorem I. Hence $\bar{v}(s)$ has a zero on the interior of the interval between two consecutive zeros of $u(s)$. Combining this result with that in theorem I we have the following fundamental theorem of comparison:*

THEOREM II. *Let u_i and v_i be solutions of* (1.4) *and* (1.5), *respectively, and let $u(s)$ and $v(s)$ denote the functions into which they interpolate linearly with respect to a given system of coördinates. If $u(s)$ has consecutive zeros on the μth and $(m+1)$th intervals, $\mu < m$, then $v(s)$ has a zero between these zeros of $u(s)$ provided that either*

(*a*) $\varphi_i \leqq \psi_i$, $i = \mu, \mu+1, \cdots, m$, *the equality sign not holding for all these values; or,*

(*b*) $\varphi_i = \psi_i$, $i = \mu, \mu+1, \cdots, m$, *and the sets of constants u_i and v_i, for $i = \mu, \mu+1, \cdots, m$, are linearly independent.*

By means of this theorem we can readily establish other properties of comparison for $u(s)$ and $v(s)$. *Suppose that $u_1 \neq 0$, $v_1 \neq 0$, $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, \nu$, and that $u_2/u_1 > v_2/v_1$. Then, if $u(s)$ has k zeros on the first ν intervals of the coördinate system, $v(s)$ has at least k zeros on these intervals and the jth of these zeros (in increasing order) of $v(s)$ is to the left of the jth one of $u(s)$.* In view of theorem II it is sufficient to prove that the first zero of $v(s)$ is to the left of the first zero of $u(s)$. Let the $(m+1)$th interva be the one containing the first zero of $u(s)$. If $m = 0$ we employ the

* Compare the related theorem due to M. B. Porter, *Annals of Mathematics* (2), 3 (1902), p. 65.

relation $u_2/u_1 > v_2/v_1$ to show readily that $v(s)$ has a zero to the left of the first zero of $u(s)$. If $m > 0$ we employ (1.6) for $\mu = 1$. Either $v(s)$ vanishes to the left of the first zero of $u(s)$ or, on taking u_1 and v_1 positive (as we may do in proof without loss of generality), we have $u_{m+2}v_{m+1} - u_{m+1}v_{m+2} > 0$, so that $v(s)$ has a zero on the $(m+1)$th interval to the left of that of $u(s)$ on this interval. This establishes the property in consideration.

Let u_1, v_1, u_{k+1}, v_{k+1} be all different from zero and let $u_2/u_1 > v_2/v_1$. Let $u(s)$ and $v(s)$ have the same number (which may be zero) of roots on the first k intervals. Then we have

$$\frac{u_{k+2}}{u_{k+1}} > \frac{v_{k+2}}{v_{k+1}}, \tag{1.7}$$

provided that $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, k$. From the result in the foregoing paragraph it follows that the rightmost root of $u(s)$ on these k intervals is to the right of that of $v(s)$, if either has a root on these intervals. Taking first the case in which they have at least one such root, let μ denote the number of the interval containing the rightmost root of $u(s)$ not at its right extremity. Without loss of generality in argument, we take $u_{\mu+1}$ and $v_{\mu+1}$ to be equal and positive. Then $v_\mu > u_\mu$ so that $u_{\mu+1}v_\mu > u_\mu v_{\mu+1}$. In (1.6) we take $m = k$. Then we have $u_{k+2}v_{k+1} > u_{k+1}v_{k+2}$, which reduces to (1.7). If $u(s)$ and $v(s)$ have no root on the k intervals, we take u_1 and v_1 positive (as we may in proof without loss of generality). We employ (1.6) with $\mu = 1$ and $m = k$. We have again $u_{k+2}v_{k+1} > u_{k+1}v_{k+2}$, whence (1.7) follows at once.

In like manner certain other similar results may be obtained from theorem II by modifying the form of certain of the inequalities in hypothesis and conclusion.

2. *Transcendental Oscillation Theorems.*—It is well known* that various types of transcendental equations may be realized as limiting cases of algebraic systems. For certain of these we shall now prove the propositions into which theorem I of § 1 passes under the appropriate limiting processes. For a homogeneous linear differential equation of the second order, $y'' + py' + qy = 0$, the result is classic in the Sturm theory: the zeros of any two linearly independent real solutions separate each other throughout any interval in which p and q are real-valued single-valued continuous functions of the real variable x.

Let us consider the difference equation

$$L(x)u(x) + M(x)u(x+1) + N(x)u(x+2) = 0 \tag{2.1}$$

in which all the indicated functions are real-valued single-valued continuous

* For references see a paper entitled "Algebraic Guides to Transcendental Problems" in *The Bulletin of the American Mathematical Society*, (2) 28 (1922): pp. 179–102.

functions of the real variable x for $x \geqq \alpha$, and $L(x)$ and $N(x)$ are both of one and the same sign for $x \geqq \alpha$. Let $u_1(x)$ and $u_2(x)$ be a fundamental system of solutions of this equation. Then, if we write

$$\omega(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1(x+1) & u_2(x+1) \end{vmatrix}, \tag{2.2}$$

we have

$$\omega(x+1) = \begin{vmatrix} u_1(x+1) & u_2(x+1) \\ u_1(x+2) & u_2(x+2) \end{vmatrix} = \frac{L(x)}{N(x)}\,\omega(x),$$

the last member being gotten from the second last on replacing $u_i(x+2)$ $(i = 1, 2)$ by its value in terms of $u_i(x)$ and $u_i(x+1)$ gotten from the fact that $u_i(x)$ satisfies (2.1). It follows from this that the numbers $\omega(a)$, $\omega(a+1)$, $\omega(a+2)$, $\cdots$ are all of one sign (and hence not zero) if a is a real number not less than α and such that $\omega(a) \neq 0$.

Let us now consider the set of constants $u_i(a)$, $u_i(a+1)$, $u_i(a+2)$, $\cdots$ and let us interpolate them linearly into the function $\bar{u}_i(x)$ with respect to a system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, $a+1$, $a+2$, $\cdots$. Then from theorem I of § 1 and the remark following it we see that the zeros of the functions $\bar{u}_1(x)$ and $\bar{u}_2(x)$ separate each other throughout the range $a \leqq x < \infty$. Let the zeros of $\bar{u}_i(x)$ on the range $a \leqq x < \infty$ be called the *characteristic points* x for the function $u_i(x)$ with respect to the point a. Then a characteristic point is a point on a sub-interval $(a+k, a+k+1)$ of the coördinate system in which $u_i(x)$ has a zero, and in fact an odd number of zeros if it has no zero at an extremity of this interval, a zero being counted an odd or an even number of times according as the function does or does not change sign in the neighborhood of the zero. If $u_i(x)$ has a zero at one end of the interval $(a+k, a+k+1)$, it does not have a zero at the other end (since $\omega(a+k) \neq 0$) and the one zero at the end is in this case a characteristic point.

The results thus obtained may be put into the form of the following theorem:

THEOREM I. *Let $u_1(x)$ and $u_2(x)$ be a fundamental system of real-valued single-valued continuous solutions of equation* (2.1) *and let $\omega(x)$ be defined by* (2.2). *Let a be a real number not less than α for which $\omega(a) \neq 0$. Then the characteristic points x for the function $u_1(x)$ and those for the function $u_2(x)$, both with respect to a, separate each other; that is, between two consecutive characteristic points for one of these functions there is one and only one for the other function.**

* Compare the closely related theorem due to M. B. Porter, *Annals of Mathematics*, (2), 3 (1902), p. 65.

This result can be readily generalized to an extensive class of functional equations of the second order. Let us consider the substitution s,

$$x' = s_x,$$

and denote by $x' = s_x^n$ the nth power of this substitution. Let it be such that there exists an open interval I of the real axis of such sort that

$$\lim_{n=\infty} s_{x_0}^n = \beta$$

for every x_0 of I, β being an end-point of I and the limit being approached monotonically. [In the case of equation (2.1) we have $s_x \equiv x + 1$ while $\alpha < x < \infty$ is a suitable open interval I for every real value of α.] *Then consider the functional equation*

$$L(x)u(x) + M(x)u(s_x) + N(x)u(s_x^2) = 0 \tag{2.3}$$

in which the functions L, M, N are real-valued single-valued continuous functions of the real variable x on the interval I and $L(x)$ and $N(x)$ are both of one and the same sign on this interval. Suppose furthermore that s_x is such that equation (2.3) *has a fundamental system of solutions $u_1(x)$, $u_2(x)$ which are real-valued single-valued and continuous on I.*

Let a be a point of I and consider the set of constants $u_i(a)$, $u_i(s_a)$, $u_i(s_a^2)$, $\cdots$; let us interpolate them linearly into the function $\bar{u}_i(x)$ with respect to the system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, s_a, s_a^2, $\cdots$. Let the zeros of $\bar{u}_i(x)$ on the part of I which is between a and β, inclusive of a and exclusive of β, be called the *characteristic points* of $u_i(x)$ with respect to a. Then, by the same procedure as in the previous case, we have the following theorem:

THEOREM II. *Let $u_1(x)$, $u_2(x)$ be a fundamental system of real-valued single-valued continuous solutions of equation* (2.3) *and let a be a point of I for which $\omega(a) \neq 0$, where*

$$\omega(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1(s_x) & u_2(s_x) \end{vmatrix}.$$

Then the characteristic points x for the function $u_1(x)$ and those for the function $u_2(x)$, both with respect to a, separate each other.

If we take $s_x \equiv qx$ where q is a real number greater [less] than unity, then a suitable interval I is that defined by the inequalities $\alpha < x < \infty$ [$\alpha > x > 0$] or that defined by the inequalities $-\alpha > x > -\infty$ [$-\alpha < x < 0$], where α is any positive number. We have therefore an interesting special case of the foregoing theorem applicable to q-difference equations of the form

$$L(x)u(x) + M(x)u(qx) + N(x)u(q^2x) = 0.$$

Each one of a great variety of functional equations finds a similar place here.

Let us next consider the analogous results for homogeneous linear

differential equations. They are classic for equations of the second order. Hence we may confine our present attention to equations of order n where $n > 2$. Such an equation we write in the general form

$$(2.4) \qquad u^{(n)} + p_1 u^{(n-1)} + \cdots + p_{n-1}u' + p_n u = 0,$$

where the superscripts refer to differentiation and where the coefficients p are real-valued single-valued continuous functions of the real variable x on the interval (ab). Since this involves an n-fold infinitude of solutions we shall require boundary conditions to restrict the permissible solutions to a two-fold infinitude so as to bring the present problem into the closest possible analogy with that involved in theorem I of § 1 and so that the new theorem shall indeed be a limiting case of that theorem. We shall suppose that these conditions are of the form

$$(2.5) \quad \sum_{j=1}^{\nu} \int_a^b L_{ij}(u)d\psi_{ij}(x) = 0, i = 1, 2, \cdots, n - 2, \quad \nu = \text{positive integer},$$

the integrals being taken in the sense of Stieltjes, the functions $\psi_{ij}(x)$ being functions of bounded variation on (ab), and the $L_{ij}(u)$ denoting homogeneous linear differential expressions in u of order not greater than $n - 1$ (the case when some $L_{ij}(u)$ are of the form $g_{ij}u$ being included by convention as differential expressions of order zero). We assume that the conditions are so chosen that (2.4) has two and just two linearly independent solutions satisfying (2.5). This is the case, for instance, when the conditions reduce to the following: $u(a)=0$, $u'(a)=0$, $\cdots$, $u^{(n-3)}(a)=0$.

Let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be a fundamental system of solutions of (2.4). Define the constants λ by means of the relations

$$(2.6) \qquad \sum_{k=1}^{\nu} \int_a^b L_{ik}(\bar{u}_j)d\psi_{ik}(x) = \lambda_{ij}, \qquad \begin{array}{l} i = 1, 2, \cdots, n-2, \\ j = 1, 2, \cdots, n. \end{array}$$

Let u be a solution of (2.4) and write it in the form

$$(2.7) \qquad u = c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_n\bar{u}_n,$$

where $c_1, c_2, \cdots, c_n$ are constants. The conditions to be met in order that u shall also satisfy conditions (2.5) may now be written in the form

$$(2.8) \qquad \lambda_{i1}c_1 + \lambda_{i2}c_2 + \cdots + \lambda_{in}c_n = 0, \qquad i = 1, 2, \cdots, n - 2.$$

The required condition that (2.4) and (2.5) shall have just two linearly independent simultaneous solutions now reduces to the condition that system (2.8), considered as a system for determining the coefficients $c_1, c_2, \cdots, c_n$, shall have just two linearly independent solutions; and for this it is a necessary and sufficient condition that the matrix $||\lambda_{ij}||$ of coefficients λ shall be of rank $n - 2$. Thus we have a necessary and sufficient condition

that the problem (2.4), (2.5) shall have two and just two linearly independent solutions.

If we write

$$u_1(x) = c_{11}\bar{u}_1 + c_{12}\bar{u}_2 + \cdots + c_{1n}\bar{u}_n,$$
$$u_2(x) = c_{21}\bar{u}_1 + c_{22}\bar{u}_2 + \cdots + c_{2n}\bar{u}_n,$$

where u_1 and u_2 are two linearly independent real solutions of (2.4), (2.5) and the c's are constants, then we have

$$\lambda_{i1}c_{11} + \lambda_{i2}c_{12} + \cdots + \lambda_{in}c_{1n} = 0, \qquad i = 1, 2, \cdots, n-2,$$
$$\lambda_{i1}c_{21} + \lambda_{i2}c_{22} + \cdots + \lambda_{in}c_{2n} = 0, \qquad i = 1, 2, \cdots, n-2.$$

It may be shown that a constant C exists, different from zero, such that

$$\begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix} = CM_{ij},$$

where M_{ij} is the algebraic complement of

$$\begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}_i' & \bar{u}_j' \end{vmatrix}$$

in the expansion of the determinant

$$D(x) \equiv \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}_1' & \bar{u}_2' & \cdots & \bar{u}_n' \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2,\,1} & \lambda_{n-2,\,2} & \cdots & \lambda_{n-2,\,n} \end{vmatrix}.$$

Now if we write

$$\omega(x) \equiv \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$$

we have

$$\omega(x) \equiv \begin{vmatrix} c_{11}\bar{u}_1 + \cdots + c_{1n}\bar{u}_n & c_{21}\bar{u}_1 + \cdots + c_{2n}\bar{u}_n \\ c_{11}\bar{u}_1' + \cdots + c_{1n}\bar{u}_n' & c_{21}\bar{u}_1' + \cdots + c_{2n}\bar{u}_n' \end{vmatrix}$$
$$\equiv \sum_{\substack{i,j=1 \\ i<j}}^{n} \begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix} \cdot \begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}_i' & \bar{u}_j' \end{vmatrix}$$
$$\equiv \sum_{\substack{i,j=1 \\ i<j}}^{n} CM_{ij} \begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}_i' & \bar{u}_j' \end{vmatrix} \equiv CD(x).$$

From this it follows that the zeros of $\omega(x)$ on (ab) are the same as those of $D(x)$ on (ab).*

* This result can be generalized to the case in which the range of i in (2·5) is over the set 1, 2, $\cdots$, k, where k is any one of the numbers 1, 2, $\cdots$, $n-1$, these conditions being so chosen that the number of linearly independent solutions of (2·4) subject to the new conditions is $n-k$. If such linearly independent solutions are denoted by u_1, u_2, $\cdots$, u_{n-k} and if constants λ are defined as before by means of a fundamental system of solutions $\bar{u}_1$, $\bar{u}_2$, $\cdots$, $\bar{u}_n$ of (2·4), then for a suitable non-zero constant C we have the identity

Let us consider any second fundamental system of solutions of (2.4). The functions in it can be represented in the form

$$\gamma_{i1}\bar{u}_1 + \gamma_{i2}\bar{u}_2 + \cdots + \gamma_{in}\bar{u}_n, \qquad i = 1, 2, \cdots, n,$$

where the γ's are constants such that the determinant $|\gamma_{ij}|$ is different from zero. If we form the determinant $D_1(x)$ of this new system of solutions, corresponding to $D(x)$ for the former system, it is easy to show by means of the multiplication theorem for determinants that we have $D_1(x) = |\gamma_{ij}| D(x)$. Hence the zeros of $D(x)$ in (ab) are independent of the fundamental system of solutions employed. The points a, b and these zeros therefore define a division of the interval (ab) into sub-intervals $I_1, I_2, I_3, \cdots$ such that $D(x)$ vanishes at the extremities of these segments other than the points a and b and does not vanish in the interior of any of these segments; and this division of (ab) into sub-intervals depends solely on the equation (2.4) and the boundary conditions (2.5).

We may now readily prove the following theorem:

THEOREM III. *If u_1 and u_2 are any two linearly independent real solutions of* (2.4), (2.5), *then between any two consecutive zeros of one of these solutions in the interior of one of the intervals $I_1, I_2, I_3, \cdots$ lies one and only one zero of the other solution.*

Since $\omega(x) \equiv CD(x)$ the determinant $\omega(x)$ is of one sign in the interior of an interval I_k. Without loss of generality we may take it to be positive in the interior of I_k; doing this we have

$$u_1u_2' - u_1'u_2 > 0 \tag{2.9}$$

within I_k. Let α and β ($\alpha < \beta$) be two consecutive zeros of u_1 on I_k. We may (and we will) take u_1 to be positive in the interior of the interval $(\alpha\beta)$, since if it were negative we could replace u_1 and u_2 by $-u_1$ and $-u_2$ without disturbing any other assumed properties or relations. Then $u_1'(\alpha) > 0$ and $u_1'(\beta) < 0$. Then, since $u_1(\alpha) = 0 = u_1(\beta)$, it follows from (2.9) that $u_2(\alpha) < 0$ and $u_2(\beta) > 0$. Hence there is one zero of $u_2(x)$ between two consecutive zeros of $u_1(x)$ in the interior of I_k. There cannot be more than one, since the result just obtained can be used to show that a zero of $u_1(x)$

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_{n-k} \\ u_1' & u_2' & \cdots & u_{n-k}' \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-k-1)} & u_2^{(n-k-1)} & \cdots & n_{n-k}^{(n-k-1)} \end{vmatrix} \equiv C \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}_1' & \bar{u}_2' & \cdots & \bar{u}_n' \\ \cdot & \cdot & \cdot & \cdot \\ \bar{u}_1^{(n-k-1)} & \bar{u}_2^{(n-k-1)} & \cdots & \bar{u}_n^{(n-k-1)} \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{k1} & \lambda_{k2} & \cdots & \lambda_{kn} \end{vmatrix}.$$

It is possible to use this result to obtain certain generalizations of theorem III and of the consequences which result from it; but when $k < n - 2$ the results do not maintain an elegance comparable with that when $k = n - 2$.

lies between any two consecutive zeros of $u_2(x)$ on I_k. Hence the theorem is established.

Let us now consider the problem of applying the result in the foregoing theorem to the case of any two linearly independent solutions of (2.4) without reference to any preassigned boundary conditions. Let u_1 and u_2 denote any two linearly independent solutions of (2.4). Holding these solutions fixed let us consider them and the differential equation in connection with certain *associated boundary conditions* of the form (2.5). By conditions *associated* with the given differential equation and given solutions we shall mean any $n-2$ conditions capable of expression in the form (2.5) and such that the solutions of (2.4) and the determined conditions (2.5) are those functions and those alone which are linearly dependent upon u_1 and u_2.

As a simple example of such associated boundary conditions we have those determined as follows: Let the initial constants for u_1 and u_2 at a point $x = \alpha$ of (ab) be

$$u_i^{(k)}(\alpha) = \rho_{ik}, \qquad k = 0, 1, 2, \cdots, n-1, \qquad i = 1, 2.$$

Let the coefficients σ_{ij} be so chosen that the equations

$$(2.10) \quad \sigma_{i0}u(\alpha) + \sigma_{i1}u'(\alpha) + \cdots + \sigma_{i,\,n-1}u^{(n-1)}(\alpha) = 0, \qquad i = 1, 2, \cdots, n-2$$

have those solutions and those alone which may be written in the form

$$u^{(k)}(\alpha) = a_1\rho_{1k} + a_2\rho_{2k}, \qquad k = 0, 1, \cdots, n-1,$$

where a_1 and a_2 are arbitrary constants. Then the solutions of (2.4) and (2.10) are those functions and those alone which are linearly dependent upon u_1 and u_2. Such conditions (2.10) afford a special form of boundary conditions to be associated with the equation (2.4) and the given linearly independent solutions u_1 and u_2.

By means of the so determined boundary conditions (2.5) and a fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ of solutions of (2.4) we may define a determinant $D(x)$, as in the earlier treatment when the conditions (2.5) were preassigned. Let the zeros of $D(x)$ on (ab) be called the *special points** of (ab) with respect to the given solutions u_1 and u_2 and the named associated boundary conditions. These special points are independent of the particular fundamental system of solutions used in defining them. By aid of the foregoing theorem we now have readily the following theorem:

THEOREM IV. *If u_1 and u_2 are any two linearly independent solutions of (2.4), then between any two consecutive zeros α, β of u_1 on (ab) there is one*

* It is clear that the special points are not altered if u_1 and u_2 are replaced by any two linearly independent functions which are linearly dependent upon them.

and just one zero of u_2 or there is on the interval $(\alpha\beta)$ a special point of (ab) with respect to the given solutions u_1 and u_2 and any set of boundary conditions associated with (2.4) and the given solutions u_1 and u_2 in the way indicated.

As a very simple example illustrating the first of the two foregoing theorems let us consider the equation and condition

$$u'''(x) + u'(x) = 0, \qquad u''(0) + u(0) = 0.$$

A fundamental system of solutions of the differential equation is 1, sin x, cos x. The determinant $D(x)$ formed with these is now

$$D(x) \equiv \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 1 & 0 & 0 \end{vmatrix} \equiv 1.$$

Hence there are no special points. Hence any two linearly independent solutions of the given differential equation satisfying the given boundary conditions have together the property that their zeros separate each other.

In illustration of the last foregoing theorem, let us consider the two solutions

$$u_1 = a + \sin x, \qquad u_2 = \cos x$$

of the same differential equation $u'''(x) + u'(x) = 0$, where a is a given real constant. We have

$$u_1(0) = a, \quad u_1'(0) = 1, \quad u_1''(0) = 0; \quad u_2(0) = 1, \quad u_2'(0) = 0, \quad u_2''(0) = -1.$$

As a single associated boundary condition relevant in this case we may take

$$u(0) - au'(0) + u''(0) = 0.$$

Then with the fundamental system 1, sin x, cos x of solutions we find that $D(x)$ has the value

$$D(x) \equiv \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 1 & -a & 0 \end{vmatrix} \equiv -(a \sin x + 1).$$

Then the corresponding special points are the zeros of a sin $x + 1$. If $|a| < 1$ they are absent and the zeros of u_1 and u_2 separate each other. If $|a| \geqq 1$ the special points interfere with this characteristic relative distribution of the zeros of u_1 and u_2. If $|a| > 1$ it is clear that u_1 has no zeros so that in this case at least one special point must occur on every interval containing two zeros of u_2.

We have assumed that the differential equation (2.4) is of order greater than 2. It is clear that a similar and in fact much simpler analysis of the same general sort as that given above is applicable to the case when $n = 2$.

and that the principal theorem which thus results is the classic theorem of Sturm. The two foregoing theorems may be considered as the natural extension of this in one important direction. Some elements of elegance are absent from the more general results, a fact which is inevitable as the simplest considerations show and as is implicitly apparent from the examples just given. The present general results afford fairly definite information as to the way in which the situation is complicated when $n > 2$.

Let $s_n(x)$ denote either $x + n$ or $q^n x$, where q is a real number greater than unity, and let us consider the functional equation of order n greater than 2,*

$$(2.11) \quad u\{s_n(x)\} + p_1 u\{s_{n-1}(x)\} + \cdots + p_{n-1} u\{s_1(x)\} + p_n u\{x\} = 0,$$

in which the coefficients $p_1, p_2, \cdots, p_n$ are real-valued single-valued continuous functions of the real variable x for $x \geqq \alpha$ (α being positive for the case of the q-difference equation). Let $u_1(x)$ and $u_2(x)$ be two real-valued single-valued continuous solutions of (2.11) which are linearly independent with respect to periodic multipliers $P(x)$ such that $P\{s_1(x)\} \equiv P(x)$. Let a be a point such that $a \geqq \alpha$ and $\omega(a) \neq 0$, where

$$\omega(x) \equiv \begin{vmatrix} u_1\{x\} & u_2(x) \\ u_1\{s_1(x)\} & u_2\{s_1(x)\} \end{vmatrix}.$$

In connection with these solutions we consider *associated boundary conditions* which are capable of representation by means of Stieltjes integrals in the form

$$(2.12) \qquad \int_a^b u\,d\psi_i(x) = 0, \qquad i = 1, 2, \cdots, n-2,$$

where b is any conveniently chosen number greater than a and the $\psi_i(x)$ are functions of bounded variation in (ab), these boundary conditions being selected in such a way that the problem (2.11), (2.12) has for real-valued single-valued continuous solutions those functions and those alone for which it is true that

$$(2.13) \quad u\{s_k(a)\} = c_1 u_1\{s_k(a)\} + c_2 u_2\{s_k(a)\}, \qquad k = 0, 1, 2, \cdots,$$

c_1 and c_2 being arbitrary constants. A particular set of such boundary conditions may be written in the form

$$\sigma_{i0} u\{a\} + \sigma_{i1} u\{s_1(a)\} + \cdots + \sigma_{i,\,n-1} u\{s_{n-1}(a)\} = 0, \qquad i = 1, 2, \cdots, n-2,$$

* The results in the remainder of this section can be extended without difficulty to a class of functional equations obtained from those treated here by replacing $s_n(x)$ by s_x^n where $x' = s_x^n$ is the nth power of the substitution $x' = s_x$ treated in connection with the second theorem of this section. Moreover, in the case of a q-difference equation in which q is positive and less than unity we may employ the transformation $x = 1/t$, $u(x) = v(t)$ and so obtain a new q-difference equation included in the class treated in the text. Similarly, on replacing x by $-t$ one changes from a q-difference equation with negative q to one with positive q.

where the σ_{ij} are so chosen that this algebraic system has those solutions and those alone which may be written in the form of the first n equations (2.13), c_1 and c_2 being arbitrary constants.

Let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be a fundamental system of solutions of (2.11) and let constants λ be defined by means of the relations

$$(2.14) \qquad \int_a^b \bar{u}_j d\psi_i(x) = \lambda_{ij}, \qquad \begin{array}{l} i = 1, 2, \cdots, n-2, \\ j = 1, 2, \cdots, n. \end{array}$$

Then if we write

$$D(x) \equiv \begin{vmatrix} \bar{u}\{x\} & \bar{u}_2\{x\} & \cdot\cdot & \bar{u}_n(x) \\ \bar{u}_1\{s_1(x)\} & \bar{u}_2\{s_1(x)\} & \cdot\cdot & \bar{u}_n\{s_1(x)\} \\ \lambda_{11} & \lambda_{12} & \cdot\cdot & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdot\cdot & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2,\,1} & \lambda_{n-2,\,2} & \cdot\cdot & \lambda_{n-2,\,n} \end{vmatrix},$$

we may readily show that $\omega\{s_k(a)\} = CD\{s_k(a)\}$, where C is independent of k, the method of proof being similar to that by which the relation $\omega(x) \equiv CD(x)$ was proved for the foregoing problem in differential equations.

By a *characteristic point* of any function $v(x)$ with respect to a we shall mean a zero (to the right of or at a) of the function $\bar{v}(x)$ obtained by linear interpolation from the constants $v\{a\}$, $v\{s_1(a)\}$, $v\{s_2(a)\}$, $\cdots$ with respect to a set of axes obtained by drawing perpendiculars to the x-axis through the points a, $s_1(a)$, $s_2(a)$, $\cdots$. The characteristic points of $D(x)$ with respect to a we shall call the *special points* for the solutions $u_1(x)$ and $u_2(x)$ with respect to the boundary conditions (2.12). They are clearly independent of the choice of fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ used in defining them.

The quantities $\omega\{s_k(a)\}$ are all of one sign if the variation of k is so restricted that all the points $s_k(a)$ lie on a single interval containing no special points. Then through use of the method employed in proving the first two theorems in this section we are led easily to the following theorem:

THEOREM V. *If u_1 and u_2 are any two real-valued single-valued continuous solutions of* (2.11) *which are independent, then between any two consecutive characteristic points α, β for u_1 with respect to a point a for which $\omega(a) \neq 0$ there is one and just one characteristic point for u_2 or there is on the interval $(\alpha\beta)$ a special point with respect to the given solutions u_1 and u_2 and any set of boundary conditions associated with* (2.11) *and the given solutions u_1 and u_2 in the way indicated.*

By taking $s_k(x) \equiv x + k$, we have in this theorem the extension, to equations of order greater than two, of the results stated for a second order difference equation in the first theorem of this section; and by taking

$s_k(x) \equiv q^k x$, $q > 1$, we have a corresponding theorem for q-difference equations.

3. *Transcendental Comparison Theorems.*—The results stated in theorem II of § 1 and the following paragraphs have as limiting cases certain of the classic theorems of comparison of Sturm for homogeneous linear differential equations of the second order; and they were indeed suggested by these Sturmian theorems. The latter, with considerable loss of elegance, have been extended to homogeneous linear differential equations of general order k.* It is possible to extend the algebraic results of the latter part of § 1 to the analogous algebraic case, namely, the case of algebraic systems with k linearly independent solutions; but the results lack (in some respects) the desired elegance. From them one may in turn obtain corresponding properties of a certain class of functional equations. We content ourselves with giving these results for the most interesting case, namely, that in which the equations have just two independent solutions.

Let us consider the functional equations

$$u(s_x^2) + \varphi(x)u(s_x) + u(x) = 0, \tag{3.1}$$

$$v(s_x^2) + \psi(x)v(s_x) + v(x) = 0, \tag{3.2}$$

in which $\varphi(x)$ *and* $\psi(x)$ *are real-valued single-valued continuous functions of the real variable* x *on the interior of the interval* I, *where* s_x, s_x^2, I *are defined as in the paragraph following theorem I of* § 2. If a is an interior point of the interval I, we define the *characteristic points* of a function $t(x)$ with respect to a to be the zeros on the interval $a \leqq x < \beta$ (or $a \geqq x > \beta$) of the function $t(x)$ derived from the constants $t(a)$, $t(s_a)$, $t(s_a^2)$, $\cdots$ by linear interpolation with respect to the system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, s_a, s_a^2, $\cdots$.

From theorem II of § 1 we have at once the following theorem:

THEOREM I. *Let* u *and* v *be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. If* $u(x)$ *has two consecutive characteristic points with respect to* a *on the* μ*th and* $(m+1)$*th intervals* $(\mu < m)$ *of the set of intervals whose end-points are the consecutive pairs of the sequence* a, s_a, s_a^2, $\cdots$, *then* $v(x)$ *has a characteristic point between these two characteristic points of* u *provided that either*

(*a*) $\varphi(x) \leqq \psi(x)$ *at the end-points of each of these intervals from the* μ*th to the* m*th inclusive, the equality sign not holding for all these end-points; or,*

(*b*) $\varphi(x) = \psi(x)$ *at the end-points of each of these intervals from the* μ*th to the* m*th inclusive and the two sets of constants,*

$$u(s_a^{\mu-1}),\ u(s_a^{\mu}),\ \cdots,\ u(s_a^{m-1}); \qquad v(s_a^{\mu-1}),\ v(s_a^{\mu}),\ \cdots,\ v(s_a^{m-1}),$$

are linearly independent.

* *Annals of Mathematics* (2), 19 (1918): 159–171.

This theorem affords certain properties of comparison for the relative distribution of the zeros of $u(x)$ and $v(x)$. Other related properties may be obtained similarly from the italicized results following theorem II of § 1; we state them without further elaboration of the proof.

THEOREM II. *Let u and v be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. Let us suppose that* $u(a) \neq 0$, $v(a) \neq 0$, $u(s_a)/u(a) > v(s_a)/v(a)$; *and that* $\varphi(x) \leqq \psi(x)$ *for* $x = a, s_a, s_a^2, \cdots, s_a^{\nu-1}$. *If* $u(x)$ *has k characteristic points on the* ν *intervals whose end-points are the consecutive pairs of the sequence* $a, s_a, s_a, \cdots, s_a^\nu$, *then* $v(x)$ *has at least k characteristic points on these intervals; and the jth of these characteristic points (counted from a towards* s_a^ν) *of* $v(x)$ *is nearer to a than the jth characteristic point of* $u(x)$.

THEOREM III. *Let u and v be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. Let* $u(a)$, $v(a)$, $u(s_a^k)$, $v(s_a^k)$ *be all different from zero and let* $u(s_a)/u(a) > v(s_a)/v(a)$. *Let* $u(x)$ *and* $v(x)$ *have the same number (which may be zero) of characteristic points on the k intervals whose end-points are the consecutive pairs of the sequence* $a, s_a, s_a^2, \cdots, s_a^k$. *Then we have*

$$\frac{u(s_a^{k+1})}{u(s_a^k)} > \frac{v(s_a^{k+1})}{v(s_a^k)},$$

provided that $\varphi(x) \leqq \psi(x)$ *for* $x = a, s_a, \cdots, s_a^{k-1}$.

Since the point a (in each of these theorems) has a great range of variation, the stated results give a large measure of information about the relative distribution of the zeros of $u(x)$ and $v(x)$ even though the arbitrary elements of the solutions of (3.1) and (3.2) are functions (rather than constants) restricted merely by a relation of the form $p(s_x) = p(x)$ and certain considerations of reality and continuity. They have their greatest use in yielding information concerning the solutions of a given equation (3.2) by comparing them with some simple functions which are known to be solutions of equation (3.1) with appropriate determination of the $\varphi(x)$.

4. *A Generalization of Boundary Conditions for Expansion Problems.*—By means of the differential expressions

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1}\frac{d^{n-1}u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u,$$

$$L_1(u) \equiv g_m \frac{d^m u}{dx^m} + \cdots + g_1 \frac{du}{dx} + g_0 u, \qquad m < n,$$

in which the coefficients l_k, g_k with their derivatives of all orders are real-valued single-valued continuous functions of x in an interval (ab) and l_n is positive throughout (ab) while g_m does not vanish in (ab) [and hence is

either positive or negative throughout (ab)], we define the differential equation

$$L(u) + \lambda L_1(u) = 0 \tag{4.1}$$

and its adjoint

$$M(v) + \lambda M_1(v) = 0, \tag{4.2}$$

where $M(v)$ and $M_1(v)$ are the adjoints of $L(u)$ and $L_1(u)$, respectively. We have classic identities of the form

$$vL(u) - uM(v) \equiv \frac{d}{dx}\{R(u, v)\}, \quad vL_1(u) - uM_1(v) \equiv \frac{d}{dx}\{R_1(u, v)\}. \tag{4.3}$$

The quantity

$$[R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b}$$

is a bilinear form in the two sets of variables

$$u^{(n-1)}(a),\ u^{(n-2)}(a),\ \cdots,\ u'(a),\ u(a),\ u^{(n-1)}(b),\ \cdots,\ u'(b),\ u(b)$$

and

$$v(a),\ v'(a),\ \cdots,\ v^{(n-2)}(a),\ v^{(n-1)}(a),\ v(b),\ v'(b),\ \cdots,\ v^{(n-1)}(b).$$

If this form is arranged in a square array whose columns contain the u's in the order written and whose rows contain the v's in the order written, then the matrix has the following properties: every element below the main diagonal is zero; every element in the upper right-hand fourth of the matrix is zero, the division being made by horizontal and vertical lines; the determinant of the matrix has the positive value $\{l_n(a)l_n(b)\}^n$.

This bilinear expression may be written in an infinite number of ways in the form

$$\begin{aligned}[R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b} \\ \equiv \sum_{i=1}^{2n} \{U_{1i}(u) + \lambda U_{2i}(u)\}\{V_{1i}(v) + \lambda V_{2i}(v)\},\end{aligned} \tag{4.4}$$

where the U_{1i}, U_{2i}, V_{1i}, V_{2i} are linear homogeneous functions with constant coefficients, the first two in the variables u and the last two in the variables v, such that the determinant of the linear forms $U_{1i}(u) + \lambda U_{2i}(u)$ in the variables u is independent of λ and different from zero and such that the determinant of the linear forms $V_{1i}(v) + \lambda V_{2i}(v)$ in the variables v has the same properties. Then with (4.1) we associate the boundary conditions

$$U_{1i}(u) + \lambda U_{2i}(u) = 0, \qquad i = 1, 2, \cdots, n; \tag{4.5}$$

and with the adjoint equation (4.2) the *adjoint* boundary conditions

$$V_{1i}(v) + \lambda V_{2i}(v) = 0, \qquad i = n+1, \cdots, 2n. \tag{4.6}$$

We say that the problem (4.1), (4.5) and the problem (4.2), (4.6) are each the *adjoint* of the other.*

The following theorems may now be proved in the same way as the special cases of them are proved in the article just cited:

THEOREM I. *If for* $\lambda = \bar{\lambda}$ *a solution* $\bar{u}(x)$ (*not identically zero*) *of* (4.1), (4.5) *exists, then a solution* $\bar{v}(x)$ *of* (4.2), (4.6) *also exists for* $\lambda = \bar{\lambda}$; *if* $\bar{u}(x)$ *is unique* (*except for a constant factor*), $\bar{v}(x)$ *is also unique* (*except for a constant factor*).

THEOREM II. *If* $y_1, y_2, \cdots, y_n$ *are* n *linearly independent solutions of* (4.1) *for* $\lambda = \bar{\lambda}$, *the condition that* $\bar{\lambda}$ *is a characteristic value is that the determinant*

$$\Delta = \begin{vmatrix} U_{11}(y_1) + \bar{\lambda} U_{21}(y_1) & U_{11}(y_2) + \bar{\lambda} U_{21}(y_2) & \cdots & U_{11}(y_n) + \bar{\lambda} U_{21}(y_n) \\ U_{12}(y_1) + \bar{\lambda} U_{22}(y_1) & U_{12}(y_2) + \bar{\lambda} U_{22}(y_2) & \cdots & U_{12}(y_n) + \bar{\lambda} U_{22}(y_n) \\ \cdot & \cdot & \cdot & \cdot \\ U_{1n}(y_1) + \bar{\lambda} U_{2n}(y_1) & U_{1n}(y_2) + \bar{\lambda} U_{2n}(y_2) & \cdots & U_{1n}(y_n) + \bar{\lambda} U_{2n}(y_n) \end{vmatrix}$$

shall vanish; a characteristic value $\bar{\lambda}$ *of* λ *is simple when and only when some first minor of* Δ *does not vanish.*

Here the terms *characteristic value* and *simple characteristic value* are used in the same sense as in the treatment of the special case referred to.

From the equations

$$L(u_i) + \lambda_i L_1(u_i) = 0, \qquad M(v_j) + \lambda_j M_1(v_j) = 0$$

we have

$$\{v_j[L(u_i) + \lambda_i L_1(u_i)] - u_i[M(v_j) + \lambda_i M_1(v_j)]\} + (\lambda_i - \lambda_j) u_i M_1(v_j) = 0.$$

Integrating with respect to x from a to b and simplifying by use of equations (4.3) and (4.4) and the boundary conditions

$$\begin{aligned} U_{1t}(u_i) + \lambda_i U_{2t}(u_i) &= 0, & t &= 1, 2, \cdots, n, \\ V_{1t}(v_j) + \lambda_i V_{2t}(v_j) - (\lambda_i - \lambda_j) V_{2t}(v_j) &= 0, & t &= n+1, \cdots, 2n, \end{aligned}$$

we have

$$(\lambda_i - \lambda_j) \sum_{k=n+1}^{2n} V_{2k}(v_j) \{ U_{1k}(u_i) + \lambda_i U_{2k}(u_i) \} + (\lambda_i - \lambda_j) \int_a^b u_i M_1(v_j) dx = 0.$$

From this relation and that obtained by interchanging the rôles of u and v we deduce at once the following theorem, generalizing theorem III of the preceding paper:

THEOREM III. *If* $u_i(x)$ *and* $v_j(x)$ *are solutions of* (4.1), (4.5) *and* (4.2),

* For a special case of these adjoint problems, with references to the literature, see AMERICAN JOURNAL OF MATHEMATICS, 43 (1921): 232–270.

(4.6), *respectively, the first for* $\lambda = \lambda_i$ *and the second for* $\lambda = \lambda_j$ *and if* $\lambda_i \neq \lambda_j$, *then we have*

$$\int_a^b u_i M_1(v_j)dx + \sum_{k=n+1}^{2n} V_{2k}(v_j)\{U_{1k}(u_i) + \lambda_i U_{2k}(u_i)\} = 0,$$

$$\int_a^b v_j L_1(u_i)dx + \sum_{k=1}^{n} U_{2k}(u_i)\{V_{1k}(v_j) + \lambda_j V_{2k}(v_j)\} = 0.$$

The novelty in this form of the problem is in the appearance of the parameter λ in the boundary conditions. It is clear that a corresponding generalization exists for the various boundary value and expansion problems treated in the memoir mentioned above; and that the development of the theory follows lines closely parallel to the earlier treatment.

5. *Expansion Problems for q-Difference and Integro-q-Difference Equations.*—Let us consider the adjoint systems of q-difference equations

$$(5.1) \qquad u_i(qx) - u_i(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda\psi_{ij})u_j(x), \qquad i = 1, 2, \cdots, n,$$

$$(5.2) \qquad v_i(x) - v_i(qx) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda\psi_{ji})v_j(qx), \qquad i = 1, 2, \cdots, n,$$

where q is a constant whose absolute value is different from unity and where the φ_{ij} and ψ_{ij} are functions of x which are analytic at infinity and have a zero there. These systems of equations possess fundamental systems*

$$u_{1j}(x),\ u_{2j}(x),\ \cdots,\ u_{nj}(x); \qquad v_{1j}(x),\ v_{2j}(x),\ \cdots,\ v_{nj}(x)$$

of solutions each function of which is analytic at infinity, say, analytic for $|x| \geqq R$, R being an appropriately chosen positive constant; moreover, the constant term in the solution $u_{ij}(x)$, and that in the solution $v_{ij}(x)$ as well, is δ_{ij} where δ_{ij} denotes unity or zero according as j is or is not equal to i. Any solution which is analytic at infinity is made up from the foregoing solutions by taking linear combinations of them with constant coefficients. We confine attention to such solutions of (5.1) and (5.2) as are analytic for $|x| \geqq R$.

If we multiply (5.1) member by member by $v_i(qx)$ and (5.2) by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$(5.3) \qquad \sum_{i=1}^{n} \delta\{u_i(x)v_i(x)\} = 0,$$

where δ denotes the operation defined by the relation $\delta f(x) \equiv f(qx) - f(x)$. Let a be a number such that $|a| \geqq R$. In (5.3) sum as to x from a to ∞,

* These existence theorems are readily proved by means of the theory of power series. There also exists a like theory in which the point zero plays the rôle played here by the point infinity.

where x runs over the values $a, qa, q^2a, \cdots$ or the values $a, q^{-1}a, q^{-2}a, \cdots$ according as $|q|$ is greater than or less than unity; thus we have

$$\sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} = 0. \tag{5.4}$$

The first member of this equation is the analogue, for the present theory, of the bilinear form in the first member of equation (4.4). It is a bilinear form in the two sets of $2n$ variables each

$$u_1(\infty),\ u_2(\infty),\ \cdots,\ u_n(\infty),\ u_1(a),\ \cdots,\ u_n(a);$$
$$v_1(\infty),\ v_2(\infty),\ \cdots,\ v_n(\infty),\ v_1(a),\ \cdots,\ v_n(a).$$

Since the bilinear form is obviously of rank $2n$ it may be written in an infinite number of ways in the normal form

$$\sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} \equiv \sum_{i=1}^{2n} U_i(u)V_i(v), \tag{5.5}$$

where the $U_i(u)[V_i(v)]$ are homogeneous linear expressions (with constant coefficients) in the variables $u[v]$. Then with our systems of q-difference equations we associate the boundary conditions

$$U_i(u) = 0, \qquad i = 1, 2, \cdots, n; \tag{5.6}$$
$$V_i(v) = 0, \qquad i = n+1, \cdots, 2n. \tag{5.7}$$

Then relation (5.4) is satisfied in virtue of the boundary conditions alone.

THEOREM I. *If for $\lambda = \bar{\lambda}$ a solution $\bar{u}_i(x)$ (not identically zero) of* (5.1), (5.6) *exists, then a solution $\bar{v}_i(x)$ of* (5.2), (5.7) *also exists for $\lambda = \bar{\lambda}$; if the solution $\bar{u}_i(x)$ is unique (except for a constant factor), the solution $\bar{v}_i(x)$ is also unique (except for a constant factor).*

For $\lambda = \bar{\lambda}$ we have

$$U_1(\bar{u}) = U_2(\bar{u}) = \cdots = U_n(\bar{u}) = 0, \qquad U_{n+k}(\bar{u}) \neq 0,$$

the inequality holding for some k of the set $1, 2, \cdots, n$, since from the relations $U_i(\bar{u}) = 0$ for $i = 1, 2, \cdots, 2n$ it would follow that $\bar{u}_i(\infty) = 0$ for $i = 1, 2, \cdots, n$, so that $\bar{u}_i$ would be identically zero, contrary to hypothesis. In the n-fold totality of solutions of (5.2) for $\lambda = \bar{\lambda}$, there is at least one, say $\bar{v}_i(x)$, which satisfies the $n - 1$ conditions

$$V_{n+j}(\bar{v}) = 0,$$

where j runs over all the numbers of the set $1, 2, \cdots, n$ except k. But (5.4) must be satisfied when u_i and v_i are replaced by $\bar{u}_i$ and $\bar{v}_i$, respectively, since the latter are taken for the same value of λ. Thence, through aid of the identity (5.5) and the boundary conditions for $\bar{u}_i$ and $\bar{v}_i$ already verified, we have $U_{n+k}(\bar{u})V_{n+k}(\bar{v}) = 0$. Therefore $V_{n+k}(\bar{v}) = 0$, so that the solution $\bar{v}_i(x)$ satisfies all the boundary conditions (5.7).

It remains to show that $\bar{v}_i(x)$ is unique whenever $\bar{u}_i(x)$ is unique. We shall prove this by supposing that the solution $\bar{v}_i(x)$ is not unique and then showing that the solution $\bar{u}_i(x)$ can not be unique. Since $\bar{v}_i(x)$ is now supposed to be not unique, let $\bar{v}_i(x)$ and $v_i^*(x)$ be two linearly independent solutions of (5.2) and (5.7) for $\lambda = \bar{\lambda}$. Then different numbers j and k of the set $1, 2, \cdots, n$ exist such that

$$V_j(\bar{v}) \neq 0, \qquad V_k(v^*) \neq 0.$$

If then

$$V_j(\bar{v})V_k(v^*) = V_j(v^*)V_k(\bar{v}),$$

we have

$$V_j(v^*) \neq 0, \qquad V_k(\bar{v}) \neq 0,$$

so that the functions

$$v_i(x) = V_j(v^*)\bar{v}_i(x) - V_j(\bar{v})v^*(x)$$

afford a solution of (5.2), (5.7) which is not identically zero. If

$$\begin{vmatrix} V_j(\bar{v}) & V_\alpha(\bar{v}) \\ V_j(v^*) & V_\alpha(v^*) \end{vmatrix} \tag{5.8}$$

is zero for every α of the set $1, 2, \cdots, n$, then we shall have $V_i(v) = 0$ for $i = 1, 2, \cdots, 2n$, a result which is impossible since it would imply that $v_i(\infty) = 0$ $(i = 1, 2, \cdots, n)$ and hence that $v_i(x)$ is the identically zero solution (contrary to the hypothesis about the linear independence of $\bar{v}_i(x)$ and $v_i^*(x)$). Hence for some α of the set $1, 2, \cdots, n$ the determinant (5.8) is different from zero. We fix upon such an α. Now choose $u_i^*(x)$, linearly independent of $\bar{u}_i(x)$, so as to satisfy the $n - 2$ conditions

$$U_i(u^*) = 0$$

for i running over all numbers of the set $1, 2, \cdots, n$ except j and α. Then from (5.5) and the boundary conditions already satisfied we have

$$U_j(u^*)V_j(\bar{v}) - U_\alpha(u^*)V_\alpha(\bar{v}) = 0,$$
$$U_j(u^*)V_j(v^*) - U_\alpha(u^*)V_\alpha(v^*) = 0.$$

Since the determinant (5.8) is different from zero it follows from this that $U_j(u^*) = 0$, $U_\alpha(u^*) = 0$, so that $u_i^*(x)$ satisfies (5.1) and (5.6) for $\lambda = \bar{\lambda}$, contrary to the hypothesis that these equations have unique solutions. Hence follows the truth of the second statement in the foregoing theorem.

A value of λ for which the system (5.1), (5.6) [and hence the system (5.2), (5.7)] has a solution will be called a characteristic value. The characteristic value is said to be simple if the solution corresponding to it is unique (except for a constant factor).

THEOREM II. *If*

are n linearly independent solutions of (5.1) *for* $\lambda = \bar{\lambda}$, *the condition that* $\bar{\lambda}$ *is a characteristic value is that the determinant*

$$\Delta = \begin{vmatrix} U_1(y^{(1)}) & U_1(y^{(2)}) & \cdots & U_1(y^{(n)}) \\ U_2(y^{(1)}) & U_2(y^{(2)}) & \cdots & U_2(y^{(n)}) \\ \cdot & \cdot & \cdot & \cdot \\ U_n(y^{(1)}) & U_n(y^{(2)}) & \cdots & U_n(y^{(n)}) \end{vmatrix}$$

shall vanish; a characteristic value $\bar{\lambda}$ *of* λ *is simple when and only when some first minor of* Δ *does not vanish.*

If we take the general solution of (5.1) for $\lambda = \bar{\lambda}$ in the form

$$u_i(x) = \sum_{j=1}^{n} c_{ij} y_i^{(j)}(x), \qquad i = 1, 2, \cdots, n,$$

we see that the vanishing of Δ is a sufficient condition that the boundary conditions (5.6) may be satisfied through an appropriate choice of the constants c_{ij}. When some first minor does not vanish this choice is unique (except for a factor which is constant through the set).

THEOREM III. *If* $u_i^{(k)}(x)$ *and* $v_i^{(l)}(x)$ *are solutions of* (5.1), (5.6) *and* (5.2), (5.7), *respectively, the first for* $\lambda = \lambda_k$ *and the second for* $\lambda = \lambda_l$ *and if* $\lambda_k \neq \lambda_l$, *then*

$$\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) u_i^{(k)}(at^s) v_j^{(l)}(at^{s+1}) = 0, \tag{5.9}$$

where $t = q$ *or* q^{-1} *according as* $|q|$ *is greater than or less than unity.*

We have the systems

$$u_i^{(k)}(qx) - u_i^{(k)}(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda_k \psi_{ij}) u_j^{(k)}(x),$$

$$v_i^{(l)}(x) - v_i^{(l)}(qx) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda_k \psi_{ji}) v_j^{(l)}(qx) - (\lambda_k - \lambda_l) \sum_{j=1}^{n} \psi_{ji}(x) v_j^{(l)}(qx);$$

multiplying the first by $v_i^{(l)}(qx)$ and the second by $-u_i^{(k)}(x)$, adding member by member, summing as to i from 1 to n and then as to x from a to infinity over the set $a, ta, t^2a, \cdots$, we have a result which (in view of the boundary conditions and the omission of the non-zero factor $\lambda_k - \lambda_l$) reduces to the relation given in the theorem.

Let us now suppose that the problem is set up so that we have the infinitude of characteristic values $\lambda_1, \lambda_2, \lambda_3, \cdots$ and corresponding solutions of the u-problem and of the v-problem; and let us suppose that the first member of (5.9) is different from zero when l and k are equal. Then if given functions $f_i(x)$ $(i = 1, 2, \cdots, n)$ have expansions in the form

$$f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \qquad i = 1, 2, \cdots, n,$$

the same coefficients c_k being employed for each of the functions, these coefficients are readily determined as follows: For fixed i multiply both members of the last equation by $\psi_{ji}(x)v_j^{(k)}(qx)$, sum as to i and j from 1 to n, and sum as to x over the set $a, ta, t^2a, \cdots$; employing theorem III we come through readily to the relations*

$$c_k = \frac{\sum_{s=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(at^s)f_i(at^s)v_j^{(k)}(at^{s+1})}{\sum_{s=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(at^s)u_i^{(k)}(at^s)v_j^{(k)}(at^{s+1})}, \qquad k = 1, 2, 3, \cdots.$$

Let us illustrate these fundamental expansion formulæ of the q-difference calculus by considering a particularly simple example. We start from the equation

$$u(qx) = \left(1 + \frac{\lambda}{x}\right)u(x), \qquad |q| < 1,$$

and its adjoint

$$\left(1 + \frac{\lambda}{x}\right)v(qx) = v(x).$$

Appropriate solutions of these are the following:

$$u(x) = 1 + \sum_{k=1}^{\infty}\frac{\lambda^k x^{-k}}{(q^{-1}-1)(q^{-2}-1)\cdots(q^{-k}-1)}, \qquad v(x) = \frac{1}{u(x)}.$$

[We may also write

$$u(x) = \prod_{k=1}^{\infty}\left(1 + \frac{q^k\lambda}{x}\right),$$

thus exhibiting $u(x)$ in factored form.] The condition (5.4) now reduces to the simpler form $u(\infty)v(\infty) - u(a)v(a) = 0$. As suitable boundary conditions implying this relation we may take

$$\alpha u(\infty) - u(a) = 0, \qquad v(\infty) - \alpha v(a) = 0,$$

where α is a given non-zero constant. Since $u(\infty) = 1 = v(\infty)$ and $u(a)v(a) = 1$, these boundary conditions require merely that λ shall satisfy the relation $u(a) = \alpha$, or

$$\alpha = 1 + \sum_{k=1}^{\infty}\frac{\lambda^k a^{-k}}{(q^{-1}-1)(q^{-2}-1)\cdots(q^{-k}-1)}.$$

The roots of this transcendental equation in λ are the characteristic values of λ for this special expansion problem. If α is not an exceptional value

* If we use a single equation of order n to replace the n equations of order unity in each system, we shall come through to an expansion problem very closely related formally to that for differential equations in § 4.

for the function of λ in the second member of this equation, the number of such characteristic values is enumerably infinite, so that we have an instance of the case supposed in the preceding paragraph. The simpler expansion formulæ pertaining to the present case may be written down at once by specializing the formulæ of the preceding paragraph.

If we apply to systems (5.1) and (5.2) a limiting process which has become classic through the investigations of Volterra, we are led through to adjoint integro-q-difference equations of the form

$$\begin{aligned} u(qx, \sigma) - u(x, \sigma) &= \int_a^\beta \{\varphi(x, \sigma, \tau) + \lambda\psi(x, \sigma, \tau)\} u(x, \tau) d\tau, \\ v(x, \sigma) - v(qx, \sigma) &= \int_a^\beta \{\varphi(x, \tau, \sigma) + \lambda\psi(x, \tau, \sigma)\} v(qx, \tau) d\tau. \end{aligned}$$

Under appropriate hypotheses the method of successive approximation yields existence theorems for equations of this sort. For them there exists also an expansion problem analogous to that which we have just treated. In the theory of this expansion problem we have the following equations analogous to (5.4) and (5.9):

$$\int_a^\beta \{u(\infty, \sigma)v(\infty, \sigma) - u(a, \sigma)v(a, \sigma)\} d\sigma = 0,$$
$$\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) u^{(k)}(at^s, \sigma) v^{(l)}(at^{s+1}, \tau) d\sigma d\tau = 0, \qquad k \neq l.$$

Then if we have an expansion of the form

$$f(x, \sigma) = \sum_{k=1}^{\infty} c_k u^{(k)}(x, \sigma),$$

the coefficients c_k have the values

$$c_k = \frac{\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) f(at^s, \sigma) v^{(k)}(at^{s+1}, \tau) d\sigma d\tau}{\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) u^{(k)}(at^s, \sigma) v^{(k)}(at^{s+1}, \tau) d\sigma d\tau}, \quad k = 1, 2, 3, \cdots.$$

A similar expansion theory exists for a great variety of generalizations of the q-difference and integro-q-difference equations of this section. It is of importance to have a detailed analysis of the character of the functions which are representable in the form of certain of these expansions.

UNIVERSITY OF ILLINOIS,
April, 1921.

ON A THEOREM IN GENERAL ANALYSIS AND THE INTERRELATIONS OF EIGHT FUNDAMENTAL PROPERTIES OF CLASSES OF FUNCTIONS.*

By E. W. Chittenden.

Introduction.

In his "Introduction to a Form of General Analysis"† E. H. Moore defined eight properties of classes of functions:

$$\mathrm{L, C, D, A, \Delta, K_1, K_2, K_{12*}}.‡$$

If we denote by $\mathfrak{P}$ a general class, Δ a development of $\mathfrak{P}$ (cf. § 1), $\mathfrak{M}$ a class of functions μ on $\mathfrak{P}$ to the class $\mathfrak{A}$ of real numbers; then the properties L, C, D, A are independent of the character of $\mathfrak{P}$, while the properties Δ, $\mathrm{K_1}$, $\mathrm{K_2}$, $\mathrm{K_{12*}}$ are defined in terms of the development Δ. The property $\mathrm{K_1}$ is a generalization of convergence, the property $\mathrm{K_2}$ of continuity, while the property $\mathrm{K_{12*}}$ is defined in terms of the functions which are at once convergent and continuous on $\mathfrak{P}$ relative to Δ and $\mathfrak{M}$. The eight properties are not independent but admit among others the important relation: $\mathrm{D\Delta K_1K_2}$ imply $\mathrm{K_{12*}}$.§

A. D. Pitcher‖ investigated the interrelations of these eight properties and their negatives and found that 108 combinations are impossible, and exhibited examples showing the possibility of 145. There remained three combinations whose character was left undetermined, out of a conceivable total of 256. His investigations led him to the conclusion that the property $\mathrm{K_{12*}}$ is implied by the composite property $\mathrm{D_1\Delta K_1K_2}B_3$, where $\mathrm{D_1}$ is a dominance property weaker than D, and B_3 is defined in terms of a development Δ.¶

The developments Δ assumed by Moore and Pitcher are finite, that is, each stage consisted of a finite number of subclasses of $\mathfrak{P}$. The author

* Presented to the American Mathematical Society at Chicago, April 21, 1916.

† The New Haven Mathematical Colloquium (Yale University Press, New Haven, 1910), pp. 1–150. To avoid frequent acknowledgments the author wishes to state that the symbolism and terminology employed are due to Moore and A. D. Pitcher (citation below).

‡ The eight properties are defined in the following pages in the order in which they occur in the discussion.

§ Moore, loc. cit., p. 145, Theorem III.

‖ "Interrelations of Eight Fundamental Properties of Classes of Functions," *The Kansas University Science Bulletin*, Vol. VII, No. 1, (1913), pp. 1–67.

¶ Pitcher, loc. cit., p. 44, Theorem I.

removed this restriction and extended the theory of MOORE to the case of infinite developments, obtaining a theorem of which the two theorems of MOORE and PITCHER mentioned are instances.*

It is the purpose of the present paper to show that the hypothesis B_3 of PITCHER is non-essential and that K_{12*} is, in fact, implied by $D_1\Delta K_1K_2$. It results from this proposition that two of the three properties whose character with respect to existence was not determined are self contradictory (cf. § 7).

§ 1. DEFINITIONS AND NOTATION.—We are concerned with classes $\mathfrak{M}$ of real and single-valued functions which are assumed to be defined for each element (point) of a given general class (space) $\mathfrak{P}$, and with properties of such classes $\mathfrak{M}$ defined in terms of a development Δ of $\mathfrak{P}$. Denoting the class of all real numbers by $\mathfrak{A}$, we speak of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

A development Δ of a class $\mathfrak{P}$ is a system $((\mathfrak{P}^{ml}))$ of subclasses $\mathfrak{P}^{ml}$ of $\mathfrak{P}$; the indices m, l being integral valued and the range of the index l, for fixed m, dependent on the value of m. The system of all classes $\mathfrak{P}^{ml}$ for a fixed value of the index m forms stage m of the development. A stage of a development is finite if the corresponding system $(\mathfrak{P}^{ml})$ is *finite*. A development is *finite* in case every stage is finite.†

Unless the contrary is stated the developments which occur in the following discussion are assumed to be finite. There will be, for each value of m, a finite number L^m ($\geqq 0$) of classes $\mathfrak{P}^{ml}$ of stage m.

If a point p belongs to a class $\mathfrak{P}^{ml}$, it is said to be *developed* of stage m. The class of all such points is denoted by $\mathfrak{P}^m$. Its complement $\mathfrak{P} - \mathfrak{P}^m$ is represented by the symbol $\mathfrak{P}^{-m}$, denoting the class of all points undeveloped of stage m.

THE RELATION K_{pm}. If a point p is undeveloped of stage m, or some later stage (that is, stage of index $m' > m$), it is in the relation K_{pm} relative to a development Δ of the space $\mathfrak{P}$. The class of all points p in the relation K_{pm} for given m is denoted by $\mathfrak{P}_m$. The complement $\mathfrak{P}_{-m}$ of $\mathfrak{P}_m$ is a very important class, in fact, the class of all points developed of stage m and every succeeding stage. The statement that p belongs to $\mathfrak{P}_{-m}$ is represented by the symbol $\overline{K}_{pm}$.

THE RELATION $K_{p_1p_2m}$. Two points p_1, p_2 of $\mathfrak{P}$ are *connected* of stage m in case there is a class $\mathfrak{P}^{ml}$ which contains both. If two points p_1, p_2 are connected of stage m or some later stage, they are in the relation $K_{p_1p_2m}$.

THE DOMINANCE PROPERTY D_1. A class $\mathfrak{M}$ of functions has the domi-

* "Infinite Developments and the Composition Property $(K_{12}B_1)_*$ in General Analysis."

† A development is *ultimately finite* in case there is an integer m_0 such that for all values of $m \geqq m_0$, stage m of the development is finite. The theory here developed is concerned only with the ultimate character of the developments which enter and there is no loss of generality in the restriction to *finite* instead of *ultimately finite* developments.

nance property D_1 in case for every pair μ_1, μ_2 of functions of $\mathfrak{M}$ there is a constant a and a function μ of $\mathfrak{M}$ such that the inequalities,

$$(1) \qquad |\mu_1| \leqq a|\mu|, \qquad |\mu_2| \leqq a|\mu|,$$

hold uniformly on $\mathfrak{P}$. [It is customary, when writing inequalities between functions which hold uniformly in the variables, to suppress the variables.]

THE PROPERTY K_2. A class $\mathfrak{M}$ has the property $K_2(K_2\mathfrak{M})$ relative to to a development Δ of the range $\mathfrak{P}$ in case for every function μ of the class $\mathfrak{M}$ there is a function μ_2 of $\mathfrak{M}$ such that *for every small positive number e there is a positive integer m_e* such that whenever two points p_1, p_2 are in the relation $K_{p_1p_2m_e}$ then

$$(2) \qquad |\mu_{p_1} - \mu_{p_2}| \leqq e|\mu_{2p_1}|.$$

[Henceforth we shall use the abbreviation: *there is an m_e*, to replace the italicized part of the preceding definition. Thus the definition of K_2 might be stated: there exists μ_2 and m_e such that $K_{p_1p_2m_e}$ implies, etc.]

§ 2. SOME CONSEQUENCES OF THE PROPERTY K_2.—If we denote the oscillation of a function μ on a class $\mathfrak{P}^{ml}$ of a development Δ of the range $\mathfrak{P}$ by $\omega^{ml}(\mu)$ and the lower bound of μ on $\mathfrak{P}^{ml}$ by $B^{ml}(\mu)$, it is easy to see from inequality (2) that whenever m exceeds m_e then

$$(3) \qquad \omega^{ml}(\mu) \leqq 2eB^{ml}(\mu_2).$$

It follows that if μ_2 vanishes on $\mathfrak{P}^{ml}$ $(m \geqq m_1)$ or has the lower bound zero, then μ is constant on $\mathfrak{P}^{ml}$. Therefore, unless μ vanishes identically on $\mathfrak{P}^{ml}$, μ is bounded from zero on that class. Conversely, if μ is not constant on $\mathfrak{P}^{ml}$, then μ_2 must be bounded from zero on $\mathfrak{P}^{ml}$. Consequently, if $\bar{\mu}$ is the common dominant of μ and μ_2, the function $\bar{\mu}$ is (in view of inequality (1) and the preceding remarks) bounded from zero on every class $\mathfrak{P}^{ml}$ $(m \geqq m_1)$. This result is stated in the following theorem:

If a class $\mathfrak{M}$ of functions on a general range $\mathfrak{P}$ with a development Δ has the property K_2 relative to Δ and the property D_1, then for every function μ of the class $\mathfrak{M}$ there is a function $\bar{\mu}$ and an integer m_1 such that, for every $m \geqq m_1$ and class $\mathfrak{P}^{ml}$, either μ vanishes identically on $\mathfrak{P}^{ml}$ or $\bar{\mu}$ is bounded from zero on $\mathfrak{P}^{ml}$.

The classes $\mathfrak{P}^{ml}$ on which μ vanishes identically form a set which may be null but is almost denumerably infinite. Relative to the function μ, we represent by $\mathfrak{P}^{ml_0}$ the classes on which μ is identically zero and by $\mathfrak{P}^{ml_1}$ the remaining classes of the development. The system Δ_0 of all classes $\mathfrak{P}^{ml_0}$ forms a development of $\mathfrak{P}$.* Likewise the system of all classes $\mathfrak{P}^{ml_1}$ forms a development Δ_1 of $\mathfrak{P}$. In the developments Δ_0, Δ_1 it is understood that

* We shall find it convenient to say that the function μ vanishes identically on the development Δ_0.

the stages $m < m_1$ are empty. For the purposes of this paper the development Δ may be regarded as equivalent to a development $\Delta_0 + \Delta_1$.

We understand by a *representative* system $((r^{ml}))$ of a development a system of points r^{ml}, where r^{ml} is an element of the class $\mathfrak{P}^{ml}$. If a class $\mathfrak{P}^{ml}$ is null, it will not have a representative element.

Let $\mathfrak{M}$ be a class with the property K_2, and suppose that p_1, p_2 are common elements of a class $\mathfrak{P}^{ml}$ of a stage $m \geqq m_e$ (effective in inequality (2)). Then if we replace p_1 by r^{ml} and let p_2 represent any element p of $\mathfrak{P}^{ml}$, we may write

$$|\mu_p| \leqq e|\mu_{2rml}| + |\mu_{rml}|.$$

If we set $e = 1$ and let a_m be the greatest value of $|\mu_{2r^{ml}}| + |\mu_{p^{ml}}|$ for all points r^{ml} representative of classes of stage m, then we have for all points of $\mathfrak{P}^m$ $(m \geqq m_1)$

$$(4) \qquad |\mu| \leqq a_m.$$

It should be noted here that the value of m_1 depends on μ.

§ 3. THE DEVELOPMENTAL PROPERTY Δ.—By definition, a class $\mathfrak{M}$ with the developmental property Δ contains a system $\mathfrak{D} = ((\delta^{ml}))$ of functions subject to two conditions of which we need consider only the first at this time: (1*a*) there is an m_e such that, for all $m \geqq m_e$,

$$(5) \qquad |\sum_g{}' \delta_p^{mg} - 1| \leqq e, \qquad |\sum_g{}' |\delta_p^{mg}| - 1| \leqq e,$$

where g denotes a value of the index l for which p belongs to $\mathfrak{P}^{ml}$, and the prime on the summation sign denotes that the summation is restricted to existent classes of stage m. The condition applies only in case p is *developed* of stage m.

Let $\mathfrak{M}$ be a class with the properties D_1, Δ. Since the development Δ is finite there are but a finite number of developmental functions δ^{ml} of each stage m. Hence from D_1, there is a function μ_m such that, for all indices l of stage m,

$$(6) \qquad |\delta_p^{ml}| \leqq |\mu_m|.$$

If we give e in inequalities (5) the value $\frac{1}{2}$, then for every point p developed of stage m there is at least one index l for which

$$(7) \qquad |\delta_p^{ml}| \geqq \frac{1}{2L^m},$$

where L^m is the number of classes $\mathfrak{P}^{ml}$ of stage m of Δ.

Therefore, the function μ_m is bounded from zero on $\mathfrak{P}^m$; in fact, for every point p developed of a stage m $(m \geqq m_{\frac{1}{2}}$, effective in inequalities

(5)), we have

$$(8) \qquad |\mu_{mp} \geqq \frac{1}{2L^m}.$$

THE PROPERTY K_1. A class $\mathfrak{M}$ has the property $K_1(K_1\mathfrak{M})$ relative to a development Δ in case for every function μ of $\mathfrak{M}$ there is a function μ_1 of $\mathfrak{M}$ and an m_e such that K_{pm_e} implies

$$(9) \qquad |\mu_p| \leqq e|\mu_{1p}|.$$

THE RELATION K_{pmm_1}. If p is a point in the relation $K_{pp_1m_1}$ with a point p_1 of $\mathfrak{P}_{-m}$ (§ 1), then p is in the relation K_{pmm_1}. The class of all such points is the class $\mathfrak{P}_{mm_1}$. It is evident that if m_1 is not less than m, $\mathfrak{P}_{mm_1} \geqq \mathfrak{P}_{-m}$.

If the class $\mathfrak{M}$ has the composite property $D_1\Delta K_1K_2$, there is for every integer m an integer $m_1 \geqq m$ such that $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$.

Suppose that the inclusion $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$ fails for every value of $m_1 > m$. Then there must exist a sequence $\{p_m\}$ of points such that for every m_1 there is a point q_{m_1} of $\mathfrak{P}_{-m}$ in the relation $K_{p_{m_1}q_{m_1}m_1}$, while p_{m_1} is not in $\mathfrak{P}_{-m_1}$, that is, the relation $K_{p_{m_1}m_1}$ holds.

Let μ be any function of the class $\mathfrak{M}$. Then, because $\mathfrak{M}$ has the property K_2, we have μ_2 and m_e such that K_{pqm_e} implies

$$(2') \qquad |\mu_p - \mu_q| \leqq e|\mu_{2q}|.$$

Since $\mathfrak{P}_{-m}$ belongs to $\mathfrak{P}^{m_1}$ for every value of $m_1 > m$, it follows from (4) that μ and μ_2 are limited on $\mathfrak{P}_{-m}$. Therefore μ and μ_2 are limited on the sequence $\{q_m\}$. Hence μ and μ_2 are limited on the sequence $\{p_m\}$.

But, from the hypothesis that the class $\mathfrak{M}$ has the property K_1, there is a function μ_1 and an integer m_e such that K_{pm_e} implies

$$|\mu_p| \leqq e|\mu_{1p}|.$$

Since μ_1 must be bounded on the sequence $\{p_m\}$ it follows from the above inequality and the hypothesis on the sequence $\{p_m\}$ that $|\mu_{p_m}|$ tends toward zero with m.

However, the class $\mathfrak{M}$ contains a function μ bounded from zero on $\mathfrak{P}^{m_1}$ for some value of $m_1 > m$. This function μ must therefore exceed some number $a_0 > 0$ at all points of $\mathfrak{P}_{-m}$ $(\leqq \mathfrak{P}^{m_1})$. But from inequality (2′) we infer, for all k such that $k \geqq m_e$, that the relation $K_{p_kq_km_e}$ holds, and

$$\begin{aligned} |\mu_{p_k}| &\geqq |\mu_{q_k}| - e|\mu_{2q_k}| \\ &\geqq a_0 - eb_2, \end{aligned}$$

where b_2 denotes the least upper bound of μ_2 on $\mathfrak{P}_{-m}$. Since e is arbitrary, the sequence $|\mu_{p_k}|$ cannot tend towards zero, contrary to the result pre-

viously obtained. Hence the hypothesis that $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$ fails for every $m_1 > m$ is untenable and the theorem is established.

From the results of § 2 and the present article we readily infer the following:

If a class $\mathfrak{M}$ has the composite property $D_1 \Delta K_2$, every function μ of $\mathfrak{M}$ is bounded on $\mathfrak{P}_{-m}$ for every value of m. For every positive integer m there is a function μ_m bounded from zero on $\mathfrak{P}_{-m}$.

Furthermore, if $\mathfrak{M}$ has the additional property K_1, then, for every positive integer m, there is an integer $m_1 > m$ such that $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$, every function μ is bounded on $\mathfrak{P}_{mm_1}$, and there is a function μ_{m_1} bounded from zero on $\mathfrak{P}_{mm_1}$.

§ 4. ON FUNCTIONS OF THE CLASS $(\mathfrak{M}'\mathfrak{M}'')_*$.—Denoting by $\mathfrak{P}'$, $\mathfrak{P}''$ two ranges which are conceptually distinct and by $\mathfrak{M}'$, $\mathfrak{M}''$ classes of functions μ', μ'' on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively, we define the class $(\mathfrak{M}'\mathfrak{M}'')_*$ as the class of all functions which are the limits of sequences of linear combinations of functions of the multiplicational composite class $\mathfrak{M}'\mathfrak{M}''$, the convergence being uniform relative to a scale function from the class $\mathfrak{M}'\mathfrak{M}''$. That is, if we introduce the notation

$$\Theta_n = \sum_{j=1}^{j=n} a_{nj}\mu'_{nj}\mu''_{nj},$$

the class $(\mathfrak{M}'\mathfrak{M}'')_*$ is the class of all functions Θ of the form

$$\Theta = L_n \Theta_n,$$

where the convergence is uniform on $\mathfrak{P}'\mathfrak{P}''$ relative to some scale function $\mu'\mu''$ of the class $\mathfrak{M}'\mathfrak{M}''$. By definition, there is an n_e such that for all $n \geqq n_e$

$$|\Theta - \Theta_n| \leqq e|\mu'\mu''|,$$

uniformly on $\mathfrak{P}'\mathfrak{P}''$.

The following proposition is an immediate consequence of this inequality:

If the classes $\mathfrak{M}'$, $\mathfrak{M}''$ each have the dominance property D_1, there exist a function $\tau_1 = \tau_1'\tau_1''$ of $\mathfrak{M}'\mathfrak{M}''$ and an integer n_1 such that, for all $n \geqq n_1$, $a\tau_1$ dominates Θ, $\mu'\mu''$, and Θ_n whenever a is a sufficiently large positive constant.

Let $\Delta_0' = ((\mathfrak{P}'^{ml_0}))$ be the system of all classes $\mathfrak{P}'^{ml_0}$ on which τ_1' vanishes identically. Then all the functions Θ, $\mu'\mu''$, Θ_n ($n \geqq n_1$) vanish identically on Δ_0. We recall, from § 2, that there exists a function σ' which dominates τ_1' and is bounded from zero on each class $\mathfrak{P}'^{ml_1}$ of the development Δ_1' whenever the class $\mathfrak{M}'$ has the properties D_1, K_2'.

The following theorem of PITCHER (loc. cit., p. 30) is of importance in the sequel:

If relative to a development Δ' of $\mathfrak{P}'$ the class $\mathfrak{M}'$ has the property K_1', and if $\mathfrak{M}'$ and $\mathfrak{M}''$ each have the dominance property D_1, then every function Θ of $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_1\mathfrak{M}'(\mathfrak{M}'')$.

§ 5. CONDITIONS IMPLYING THAT THE CLASS $(\mathfrak{M}'\mathfrak{M}'')_*$ HAVE THE PROPERTY $K_2'\mathfrak{M}'(\mathfrak{M}'')$.—MOORE has shown that if $\mathfrak{M}'$ has the property DK_{12}', and $\mathfrak{M}''$ has the property D,* the class $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$.† PITCHER showed that the property D in the hypothesis on $\mathfrak{M}'$, can be replaced by the weaker property $D_1 B_3'$. The property B_3' will be defined later. We shall show that the conclusion is true under the hypothesis, $\mathfrak{M}'$ has the property $D_1\Delta' K_1' K_2'$.

If a class $\mathfrak{M}'$ has the composite property $D_1\Delta' K_{12}'$, and a class $\mathfrak{M}''$ has the property D, every function of the class $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$.

The proof of this theorem follows the plan of the proof of the corresponding theorem of PITCHER,‡ differing from his proof in details whose character will be indicated later.

From the theorem of MOORE stated in § 4 there exists for every function Θ of $(\mathfrak{M}'\mathfrak{M}'')_*$ a scale function $\mu_1'\mu_1''$ such that $K_{pm_e'}'$ implies:

$$|\Theta_{p'}| \leqq e\,|\mu_{1p'}'\mu_1''|. \tag{10}$$

Since $\mathfrak{M}'$ has the property K_2' there is a function μ_0' and integer m_e'' such that $K_{p_1'p_2'm_e''}'$ implies:

$$|\mu_{1p_1'}' - \mu_{1p_2'}'| \leqq e\,|\mu_{0p_1'}'|. \tag{11}$$

It readily follows from the two preceding inequalities that there is an m_e such that the simultaneous presence of the relations

$$K_{p_1'm_e}', \qquad K_{p_2'm_e}', \qquad K_{p_1'p_2'm_e}'$$

implies

$$|\Theta_{p_1'}| + |\Theta_{p_2'}| \leqq e\,|\mu_{0p_1'}'\mu_1''|. \tag{12}$$

By definition, if a class $\mathfrak{M}$ has the dominance property D, there exists for every sequence $\{\mu_m\}$ of functions of $\mathfrak{M}$ a function μ and sequence $\{a_m\}$ of positive real numbers such that for every m, $|a_m\mu| \geqq |\mu_m|$. It is at once evident that any class of functions which is limited on a range $\mathfrak{Q}$ and contains a function bounded from zero on $\mathfrak{Q}$ has the property D.

The development Δ' of $\mathfrak{P}'$ determines by an obvious reduction a development Δ_m' of the class $\mathfrak{P}_{-m}'$. From the results of §§ 2, 3 it follows that

* A class $\mathfrak{M}$ has the dominance property D in case there exists for every sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ a function μ of $\mathfrak{M}$ and sequence $\{a_n\}$ of positive constants such that for every n, $|\mu_n| \leqq a_n|\mu|$.

† A function θ on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$ in case there is a function $\mu'\mu''$ of $\mathfrak{M}'\mathfrak{M}''$ and an m_e such that $K_{p_1'p_2'm_e}'$ implies

$$|\Theta_{p_1'} - \Theta_{p_2'}| \leqq e\,|\mu_{p_1'}'\mu''|.$$

A class N of functions on $\mathfrak{P}'\mathfrak{P}''$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$ in case every function Θ of N has the property.

‡ Cf. PITCHER, loc. cit., § 39, pp. 40–43.

relative to $\mathfrak{P}'_{-m}$ the class $\mathfrak{M}'$ has the property D. It is easy to see that every function has the property K'_1 relative to the development Δ'_m of $\mathfrak{P}'_{-m}$, since no element of $\mathfrak{P}'_{-m}$ is undeveloped after stage m. Hence the class $\mathfrak{M}'_m$ of functions on $\mathfrak{P}'_{-m}$, obtained by considering the functions of $\mathfrak{M}$ on $\mathfrak{P}'_{-m}$ only, has the composite property DK'_{12} relative to to Δ'_m. The result of MOORE cited above applies here, and we conclude that relative to Δ'_m the function Θ has the property $K'_2\mathfrak{M}'(\mathfrak{M}'')$. That is, there exists, for every integer m, a function $\mu'_m\mu''_m$ of $\mathfrak{M}'\mathfrak{M}''$ and an integer m_{em} such that $K'_{p'_1p'_2m_{em}}$ implies (when p'_1, p'_2 are points of P'_{-m})

$$(13) \qquad |\Theta_{p'_1} - \Theta_{p'_2}| \leqq e\,|\mu'_{mp_1}\mu''_m|.$$

We have seen in § 4 that there exist in $\mathfrak{M}'\mathfrak{M}''$ a function $\tau_1 = \tau'_1\tau''_1$ such that τ_1 dominates Θ, and that, relative to τ'_1, $\Delta' = \Delta'_0 + \Delta'_1$, where τ'_1 vanishes identically on Δ'_0, and a function σ', dominating τ'_1, which is bounded from zero on every class $\mathfrak{P}^{ml_1}$ of Δ'_1.

Since σ' is bounded from zero on the points of $\mathfrak{P}'_{-m}$ which belong to classes $\mathfrak{P}'^{ml_1}$, and since μ'_m is bounded on $\mathfrak{P}'_{-m}$, we may find a constant a'_m such that $|\mu'_m| \leqq a'_m|\sigma'|$ for all such points of $\mathfrak{P}'_{-m}$. From the property D (of the class $\mathfrak{M}''$) there is a function σ'' in $\mathfrak{M}''$ and a sequence $\{a''_m\}$ of positive constants such that, for all points of $\mathfrak{P}''$,

$$|\mu''_m| \leqq a''_m|\sigma''|.$$

Replacing $\mu'_m\mu''_m$ in inequalities (13) by $\sigma'\sigma''$ we obtain m_{em} such that $K'_{p'_1p'_2m_{em}}$ implies, whenever p'_1, p'_2 are elements of $\mathfrak{P}'_{-m}$,

$$(14) \qquad |\Theta_{p_1} - \Theta_{p'_2}| \leqq e\,|\sigma'_{p'_1}\sigma''|.$$

For, if p'_1, p'_2 are connected of stage $m' \geqq m_{em}$ of Δ', they are connected of this stage in one of the two developments Δ'_0, Δ'_1. In the first case $\Theta_{p'_1} = \Theta_{p'_2} = 0$ (p''), and in the second case

$$|\mu'_{mp'_1}\mu''_m| \leqq a'_m a''_m|\sigma'_{p'_1}\sigma''|.*$$

Let $\nu = \nu'\nu''$ be a common dominant of $\mu'_1\mu''$, $\mu'_0\mu''_1$, $\sigma'\sigma''$. Then there is an integer m_e such that

$$(15) \qquad K'_{p'_1m_e}K'_{p'_2m_e}K'_{p'_1p'_2m_e} \quad \textit{implies} \quad |\Theta_{p'_1}| + |\Theta_{p'_2}| \leqq e\,|\nu_{p'_1}|.$$

From a theorem of § 3, we have an integer m'_e (dependent on m_e) such that

* The inequality (14) can be established with less difficulty, in fact without the intervention of the system Δ_0, if the class $\mathfrak{M}'$ contains a function μ' bounded from zero on $\mathfrak{P}'_{-m}$ for every value of m. This fact led A. D. PITCHER to make this hypothesis which he introduced as the property B'_3. In the present proof the hypothesis Δ is introduced to provide functions bounded from zero on the classes $\mathfrak{P}_{-m}$. The hypothesis B_3 provides a single function effective on all $\mathfrak{P}_{-m}$. Such functions are needed in order to prove that the reduction of $\mathfrak{M}'$ relative to $\mathfrak{P}_{-m}$ has the dominance property D.

whenever $m \geqq m'_e$ then $\mathfrak{P}'_{m_e m} \leqq \mathfrak{P}'_{-m}$; and for $m = m'_e$ there is, because of inequality (14), an integer $m''_e > m'_e$ such that

$$(16) \qquad \overline{\mathrm{K}}'_{p_1 m'_e} \overline{\mathrm{K}}'_{p_2 m_e} \mathrm{K}'_{p'_1 p'_2 m''_e} \qquad \textit{implies} \qquad |\Theta_{p'_1} - \Theta_{p'_2}| \leqq e|\nu_{p'_1}|.$$

It follows, from the inclusion of $\mathfrak{P}'_{m_e m'_e}$ in $\mathfrak{P}'_{-m'_e}$, that $\mathrm{K}'_{p'_1 p'_2 m'_e}$ implies that both p'_1, p'_2 belong to $\mathfrak{P}_{-m'^e}$ or else neither is contained in $\mathfrak{P}'_{-m'_e}$. It is now easy to see that if p'_1, p'_2 are in the relation $\mathrm{K}'_{p'_1 p'_2 m''_e}$, then one of the inequalities (15), (16) must hold. That is, there is a function μ ($= \nu$) and positive integer m_e ($= m''_e$) such that

$$(17) \qquad \mathrm{K}'_{p'_1 p'_2 m_e} \qquad \textit{implies} \qquad |\Theta_{p'_1} - \Theta_{p'_2}| \leqq e|\mu_{p'_1}|,$$

which was to be proved.

§ 6. THE PROPERTY K_{12*}.—A class $\mathfrak{M}'$ has the property K'_{12*} if for every class $\mathfrak{M}''$ with the composite property LCD* the corresponding class $(\mathfrak{M}'\mathfrak{M}'')_*$ is *equal* to the class of all functions Θ on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ which are functions of $\mathfrak{M}''$ for every element p' of $\mathfrak{P}'$ and have the property $\mathrm{K}'_{12}\mathfrak{M}'(\mathfrak{M}'')$. When the class $\mathfrak{M}''$ does not enter explicitly we drop the prime and speak of the property K_{12*} of a class $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

MOORE demonstrated that the property $\mathrm{D}\Delta\mathrm{K}_1\mathrm{K}_2$ implies the property K_{12*}. PITCHER showed that K_{12*} is implied by $\mathrm{D}_1\Delta\mathrm{K}_1\mathrm{K}_2B_3$, where B_3 is defined in terms of the development Δ (cf. § 5). From the result of § 5 and the following theorem of MOORE, 'if $\mathfrak{M}'$ has the properties D_1, Δ', K'_1, K'_2, and $\mathfrak{M}''$ has the property LCD, every function which has the property $\mathrm{K}'_{12}\mathfrak{M}'(\mathfrak{M}'')$ and belongs to $\mathfrak{A}\mathfrak{M}''$ for every point p' of $\mathfrak{P}'$ belongs to $(\mathfrak{M}'\mathfrak{M}'')_*$,'† we have the theorem which forms the subject of this paper:

If $\mathfrak{M}$ is a class of functions on a range $\mathfrak{P}$ to the class $\mathfrak{A}$ of real numbers and has the properties D_1, Δ, K_1, K_2, *relative to a finite development Δ of $\mathfrak{P}$, then the class $\mathfrak{M}$ has the property* K_{12*}.

In view of related results of MOORE we obtain at once the further conclusion:

The class $\mathfrak{M}_*$ has the properties L, C, D_1, Δ, K_1, K_2, K_{12*}.

§ 7. ON THE INTERRELATIONS OF THE EIGHT PROPERTIES L, C, D, A, Δ, K_1, K_2, K_{12*}.—All of the eight properties, L, C, D, A, Δ, K_1, K_2, K_{12*}, have been defined previously except the property A. A class $\mathfrak{M}$ is absolute

* The property D is defined in a footnote to § 5. A class $\mathfrak{M}$ of functions is LINEAR (L) if it contains the function $a_1\mu_1 + a_2\mu_2$, whenever μ_1 and μ_2 are functions of $\mathfrak{M}$ and a_1, a_2 are real constants. A class $\mathfrak{M}$ is CLOSED (C) if it contains all functions Θ of the form

$$\Theta = \mathrm{L}_n\mu_n(\mathfrak{P};\mu),$$

that is, all limits of sequences of functions of $\mathfrak{M}$ converging uniformly relative to a scale function of the class $\mathfrak{M}$.

† MOORE, loc. cit., § 80, Theorem II, p. 140. The class $\mathfrak{A}\mathfrak{M}''$ is the class of all functions $a\mu''$, where a is a real number and μ'' belongs to $\mathfrak{M}''$.

(A) if it contains the absolute function $|\mu|$ of every function μ which belongs to $\mathfrak{M}$.

If a class $\mathfrak{M}$ is linear (L) and absolute (A), it has the dominance property D_1. It follows that the property $LA\Delta K_1K_2$ implies K_{12*} (§ 6). Denoting the absence of a property P by the symbol ^-P, we represent two of the three composite properties whose character was left undetermined by PITCHER as follows:

$$LC\,^-DA\Delta K_1K_2\,^-K_{12*}$$
$$L\,^-C\,^-DA\Delta K_1K_2\,^-K_{12*}.$$

It is clear that these combinations are self contradictory.

Of the 256 possible combinations of the eight properties the results of PITCHER and the present paper show that 145 exist and 110 do not exist as composite properties of a class $\mathfrak{M}$ of functions.

The remaining combination

$$LC\,^-D\,^-A\Delta K_1K_2\,^-K_{12*}$$

presents an unsolved problem. PITCHER has exhibited a number of classes of systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$ for which this composite property does not exist. From the theorem of § 6 we may add that this composite property is non-existent in any system in which the class $\mathfrak{M}$ has the property D_1.

THE UNIVERSITY OF IOWA,
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PLANE INVOLUTIONS OF ORDER FOUR.

BY TEMPLE RICE HOLLCROFT.

1. A plane involution of order four or a (1, 4) point correspondence between two planes is said to exist when to one point in the first plane correspond four points in the second to each of which corresponds the original point in the first. The planes may be distinct or coincident.

Plane involutions of order two have been completely classified by Bertini* and those of order three by Miss Howe.† Miss Scott‡ has treated metrically a case of (1, 4) point correspondence which is a special form of type 3 of this classification.

The purpose of this paper is to discuss and classify the algebraic methods of relating two planes in (1, 4) point correspondence. All the methods that are reducible to each other by means of birational transformations will be counted as one, and the most general of these will be called a type of (1, 4) point correspondence. The types are thus birationally independent.

The general methods employed are due to Sharpe and Snyder§ and have been made use of since by A. M. Howe.|| The new features of plane involutions of order four over those of order three and two are found principally in the discussion of coincident images. Involutions of order higher than four, however, present no new features, that is, an involution of order four may be generalized for any order n as demonstrated in the latter part of this paper. It is felt, therefore, that involutions of order four are of sufficient importance to warrant their classification, since they are of the highest order that has distinctive characteristics and those of order n cannot be classified.

2. A type is defined by two algebraic equations in x_1, x_2, x_3 and x'_1, x'_2, x'_3 of the form

$$x'_1u_1 + x'_2u_2 + x'_3u_3 = 0, \tag{1}$$

$$x'_1v_1 + x'_2v_2 + x'_3v_3 = 0, \tag{2}$$

* E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 2, Vol. VIII (1877), pp. 244–286.

† A. M. Howe, "A Classification of Plane Involutions of Order Three," AMERICAN JOURNAL OF MATH., Vol. XLI (1919), pp. 25–40.

‡ C. A. Scott, "Studies in Transformation of Plane Algebraic Curves," *Quarterly Journal of Mathematics*, Vol. 29 (1899), pp. 329–381, and Vol. 32 (1901), pp. 209–239.

§ F. R. Sharpe and V. Snyder, "Types of (2, 2) point correspondences between two planes," *T. A. M. S.*, Vol. 18 (1917), pp. 402–414.

|| A. M. Howe, loc. cit.

wherein $u_i = 0$ and $v_i = 0$ are curves of (x) intersecting in four variable points. These equations so relate the planes (x) and (x') that to a point P' of (x') correspond P_1, P_2, P_3, P_4 of (x) and to each of these image points of (x) corresponds the original point P', for the coördinates of a point of (x') determine uniquely a pair of curves of (x) intersecting in four non-basic points and the two curves so determined are in (1, 1) correspondence with the coördinates of the point of (x'). Ordinarily the four image points will be distinct, but the coördinates of P' may be so chosen that the corresponding curves of (x) shall have one or two contacts. The locus of such points P' is a curve of (x') called the curve of branch points. It will be denoted by the letter L'. The locus of the contacts in (x) is called the coincidence curve, denoted by K. When two image points coincide on K, the other two lie on a residual curve Γ. The complete image of K is L'; of L', $K^2\Gamma$; of Γ, L'^2. L' and K are in (1, 1) correspondence; L' and Γ (and therefore K and Γ) are in (1, 2) point correspondence.

To a line of (x') corresponds a curve of (x) whose image is the original line of (x') counted four times. To two lines of (x') intersecting at P' correspond two curves of (x) intersecting in four non-basic points, the images of P'. The remaining intersections of the two image curves of (x) are fixed points common to all image curves of (x). Since these curves are in (1, 1) correspondence with the lines of (x') they form a net. The basis points of this net are the fixed points common to all the image curves. Their images in (x') are basis curves the order of the curve being equal to the multiplicity of the point on the line images. When a given curve of (x) passes through a basis point, its image in (x') is composite, consisting of the curve-image of the basis point (counted as many times as the multiplicity of the basis point on the given curve) and a curve called the proper image of the given curve of (x).

The jacobian of the net of line image curves of (x) is the locus of the contacts of the curves of the net. It therefore either is the coincidence curve K or contains it as a factor. In the latter case the other factors are basis curves of (x). The residual curve Γ is sometimes called the cojacobian.

To the lines of (x) correspond rational curves of (x'). Since the transformation from (x') to (x) is not rational the image curves of (x') can have basis points only of higher multiplicity than two. The images of these basis points are irrational basis curves of (x) whose order is the multiplicity of the basis point on the line images of (x').

3. There are ten independent types of (1, 4) point correspondences. Each type is defined by two equations of the form described in the preceding section. The equations are both linear in (x'), so the types depend only

on the choice of u_i and v_i of (x). The following table shows the curves represented by u_i and v_i for each type. Subscripts of C denote the order of a curve, coefficients and subscripts of P the number and multiplicity respectively of its basis points.

Type	$u_i = 0$	$v_i = 0$
1	line pencil	C_n; P_{n-4}
2	line	quartic
3	cubic; $5P_1$	cubic; $5P_1$
4	conic	conic
5	cubic; $8P_1$	C_{12}; $8P_4$
6	cubic; $8P_1$	quartic; $8P_1$
7	conic; P_1	cubic; P_2
8	conic; $2P_1$	cubic; $2P_1$
9	conic; $2P_1$	quartic; $2P_2$
10	cubic; $7P_1$	quartic; P_2, $6P_1$

4. *Type 1.*—The defining equations are

$$x_1x_1' + x_2x_2' = 0, \tag{1}$$

$$x_1'\varphi_1 + x_2'\varphi_2 + x_3'\varphi_3 = 0, \tag{2}$$

$\varphi_i(x) \equiv x_3^4u_i(x_1, x_2) + x_3^3v_i(x_1, x_2) + x_3^2w_i(x_1, x_2) + x_3s_i(x_1, x_2) + t_i(x_1, x_2) = 0,$ wherein u_i, v_i, w_i, s_i, t_i are homogeneous functions of the respective degrees $n-4$, $n-3$, $n-2$, $n-1$, n in x_1, x_2. Then equation (2) represents a C_n with an $(n-4)$-fold point at $P \equiv (0, 0, 1)$, the vertex of the line pencil. The equations of transformation from (x') to (x) are

$$\rho x_1' = -x_2\varphi_3,$$
$$\rho x_2' = x_1\varphi_3,$$
$$\rho x_3' = x_2\varphi_1 - x_1\varphi_2.$$

A line of (x') with coefficients a_1', a_2', a_3' corresponds to

$$C_{n+1} \equiv (a_1'x_2 - a_2'x_1)\varphi_3 + a_3'(x_1\varphi_2 - x_2\varphi_1) = 0.$$

C_{n+1} has P_{n-3} and is of genus $3n-6$. The curves $\varphi_3 = 0$ and $x_1\varphi_2 - x_2\varphi_1 = 0$ intersect in $8n-12$ points outside of P. These $8n-12$ P_1 are simple basis points of (x) through which pass all image curves of (x). Two line image curves intersect in four non-basic points which are collinear with P and are the images of the intersection of the two lines of (x'). If the line of (x') passes through $P' \equiv (0, 0, 1)$, its image is

$$\varphi_3(a_1'x_2 - a_2'x_1) = 0.$$

The curve $\varphi_3 = 0$ is fundamental, the image of P', and the other factor is the proper image of the line through P'.

To a line of (x) with coefficients a_1, a_2, a_3 corresponds

$$C'_{n+1} \equiv x'_1\varphi'_1 + x'_2\varphi'_2 + x'_3\varphi'_3 = 0,$$

wherein φ'_i has the same form as φ_i with $-a_3x'_2$, $a_3x'_1$ and $a_1x'_2 - a_2x'_1$ substituted respectively for x_1, x_2 and x_3. C'_{n+1} has an n-fold point at P' and is of genus 0. The proper image of a line through P is a line through P' counted four times. The basis point P has for its image the basis curve

$$u' \equiv x'_1u_1(-x'_2, x'_1) + x'_2u_2(-x'_2, x'_1) + x'_3u_3(-x'_2, x'_1) = 0.$$

The basis curve u' is of order $n-3$ with P'_{n-4}.

Each of the $8n-12$ P_1 corresponds to a line of (x') through P' and tangent to L'.

5. The equation of the branch-point curve L' is the condition on the parameters x'_i that the line and C_n given by the defining equations have a point of contact. This condition is of degree $6n-12$ in the coefficients of the line and 6 in the coefficients of the C_n. L' is therefore of order $6n-6$ with P'_{6n-12}. L' will also be determined as the image of K.

The jacobian of the net of line images in (x) is

$$\varphi_3\left[\varphi_3\left(x_2\frac{\partial\varphi_1}{\partial x_3} - x_1\frac{\partial\varphi_2}{\partial x_3}\right) - \frac{\partial\varphi_3}{\partial x_3}(x_2\varphi_1 - x_1\varphi_2)\right] = 0.$$

The jacobian consists of the basis curve $\varphi_3 = 0$ and the coincidence curve K. K is of order $2n$ and genus $10n-20$ with P_{2n-6} and $8n-12$ P_1. The image of K is L'_{6n-6} with P'_{6n-12} and of genus $10n-20$. Then L' has the equivalent of $20n-30$ double points.

If in the two defining equations we solve the equations of the line and C_n simultaneously, simplify and arrange in powers of x_3/x_1, the discriminant of the quartic in x_3/x_1 is the equation of L'. Thus

$$L'_{6n-6} \equiv S^3 + 27T^2,$$

wherein

$$S \equiv 4s'v' - u't' - 3w'^2,$$
$$T \equiv u'w't' + 2v'w's' - v'^2t' - u's'^2 - w'^3;$$

u' is the fundamental curve of (x') and v', w', s', t' are curves whose equations are like that of u' with v, w, s, t substituted respectively for u. The curves S and T intersect in $12n-18$ non-basic points, which are cusps of L'. Then of the $20n-30$ double points of L' $12n-18$ are cusps and $8n-12$, nodes. The coördinates of a cusp of L' determine the line and C_n of the defining equations so that the line is tangent to the C_n at a point of inflection, and the coördinates of a double point of L' so that the line is a double tangent to the C_n.

The class of L' is $20n-36$. Then from P' $8n-12$ tangents can be drawn to L' which are the basis lines of (x').

The image of L' is a curve of order $12n - 6$ consisting of K counted twice and the residual curve Γ_{8n-6} which has P_{8n-18}, $8n - 12P_4$, $12n - 18$ cusps and is of genus $28n - 53$.

K and Γ have $40n - 60$ intersections among which are $12n - 18$ contacts, the common tangent passing through P. These contacts correspond to the cusps of L', that is, at these contacts three image points coincide. The fourth image point lies at a cusp of Γ on this common tangent. The remaining $16n - 24$ are simple intersections, images of the nodes of L'. To a node of L' correspond two intersections of K and Γ such that the line joining them is tangent to Γ at both points and passes through P. The two intersections represent a pair of coincidences.

The class of K is $24n - 42$. From P $20n - 30$ tangents can be drawn to K consisting of $8n - 12$ at the simple basis points and $12n - 18$ at the contacts of K and Γ. The class of Γ is $60n - 102$. From P $44n - 66$ tangents may be drawn to Γ which consist of $12n - 18$ at the contacts of K and Γ, $8n - 12$ bitangents through P and the $8n - 12$ P_1 and $8n - 12$ bitangents at the pairs of coincidences.

The four images of a point of (x') cannot ordinarily coincide because a general C_n of the defining equations has no point of undulation. When the equation of the C_n is so chosen that it will have such a point, certain of the pairs of coincidences will themselves coincide making four coincident images. Such a point is an undulation of Γ, the double tangent passing through P and a node of K neither of whose branches has contact with Γ. The corresponding point of L' is a cusp of the second kind.

6. L' and the basis curve u' intersect in $12n - 30$ points of which $2n - 6$ are contacts corresponding to the $2n - 6$ directions of K through P and $8n - 18$ intersections corresponding to the directions of Γ through P.

The image of the basis curve u' is P_{n-3} and a residual curve p of order $2n - 3$ with P_{2n-6} and $8n - 12$ P_1. K and p have the same tangents at P. K meets p in $10n - 24$ points besides at basis points. Of these $2n - 6$ lie on the $2n - 6$ common tangents to K and p at P and $8n - 18$ lie on the $8n - 18$ tangents to Γ at P. The class of p is $12n - 30$. From P $8n - 18$ tangents can be drawn to p. These tangents coincide with the tangents to Γ at P and pass through intersections of p with K and of p with Γ. Γ and p intersect in $16n - 42$ points of which $8n - 24$ are $4n - 12$ contacts lying two each on the $2n - 6$ common tangents to p and K at P and $8n - 18$ intersections lying on the tangents to p from P and to Γ at P. The complete image of p is u' counted three times.

The image of the point of contact of a basis line of (x') and L' consists of two consecutive points of K at the basis point and two pairs of consecutive points of Γ not at the basis point. Then the line joining P to

a P_1 has a contact with K at P_1 and two contacts with Γ not at P_1. A line through P and a P_1 has for its image the basis line of (x') corresponding to that P_1 counted three times.

7. For the remaining types only the most important results are given. The defining equations for each type may be constructed from the table in section 3. The following notation will be used in tabulating results:

The symbol $\sim$ meaning "corresponds to."

L', K, Γ, curves as heretofore described.

C, C', variable curves of (x) and (x').

P, P', basis points of (x) and (x').

$\overline{P}$, $\overline{P}'$, variable points of (x) and (x').

k, double points of L'.

Subscripts of curves denote their order.

Subscripts of points denote their multiplicity on the curve being described.

Type 2.

$C'_1 \sim C_5,\ 21P_1$.

$C_1 \sim C'_5,\ \overline{P}'_4$.

$L'_{18},\ 102k \sim K^2_{12},\ 21P_2;\ \Gamma_{66},\ 21P_{14}$.

Type 3.

$C'_1 \sim C_6,\ 5P_2,\ 12P_1$.

$C_1 \sim C'_6,\ 10\overline{P}'_2$.

$L'_{16},\ 76k \sim K^2_{15},\ 5P_5,\ 12P_2;\ \Gamma_{66},\ 5P_{22},\ 12P_{12}$.

Type 4.

$C'_1 \sim C_4,\ 12P_1$.

$C_1 \sim C'_4,\ 3\overline{P}'_2$.

$L'_{12},\ 39k \sim K^2_9,\ 12P_2;\ \Gamma_{30},\ 12P_8$.

Type 5. Q_1 is the ninth basis point of the cubic pencil.

$C'_1 \sim C_{15},\ 8P_5,\ Q_1,\ 20P_1$.

$C_1 \sim C'_{15},\ P'_{12},\ 25\overline{P}'_2$.

$Q_1 \sim$ line not through P'.

$P_1 \sim$ line through P'.

$L'_{28},\ P'_{20},\ 116k \sim K^2_{30},\ 8P_{10},\ Q_2,\ 20P_1;\ \Gamma_{120},\ 8P_{40},\ Q_{24},\ 20P_6$.

Type 6.

$C'_1 \sim C_7,\ 8P_2,\ Q_1,\ 12P_1$.

$C_1 \sim C'_7,\ P'_4,\ 9\overline{P}'_2$.

$L'_{20},\ P'_{12},\ 79k \sim K^2_{14},\ 8P_4,\ Q_2,\ 12P_1;\ \Gamma_{64},\ 8P_{20},\ Q_{16},\ 12P_6$.

Type 7.

$C'_1 \sim C_5,\ P_3,\ 12P_1$.

$C_1 \sim C'_5,\ \overline{P}'_4$.

$L'_{12},\ 40k \sim K^2_{12},\ P_8,\ 12P_2;\ \Gamma_{36},\ P_{20},\ 12P_8$.

When C_1' passes through P' the plane determined by C_1' and P_1 passes through P_n and the projection of the curve common to that plane and S from P_n on (x) is a line through P cut out by that plane. Thus the pencil of planes on P_1P_n cuts out the corresponding rays of the pencils on P and P' of (x) and (x') respectively. The $n(n+1)$ fixed planes of the pencil determined by the $n(n+1)$ lines of S cut out the $n(n+1)$ basis lines of (x') and the lines of (x) through P and the respective P_1. Only proper images appear in this construction.

A plane through a line C_1 of (x) and P_n cuts S in a curve of order $n+1$ with an n-fold point at P_n which projects from P_1 into a C'_{n+1} of (x') with P' of multiplicity n.

The tangent cone to S from P_1 is of order $(n-1)(n-2)$. From P_1 can be drawn a certain number of inflectional tangents and double tangents which are respectively cuspidal and double edges of the tangent cone. The intersection of this tangent cone with (x') gives the branch-point curve L' whose cusps and nodes are occasioned by the cuspidal and double edges of the cone. The basis lines of (x') are seen to be tangents to L'.

The projection from P_n on (x) of the contour curve common to S and the tangent cone gives the coincidence curve K. Each element of the tangent cone cuts S in $n-2$ other points besides at the ordinary contacts. The projection on (x) from P_n of these residual intersections is the curve Γ. An inflectional contact projects from P_n into three consecutive points, two on K and one on Γ, so that K and Γ touch at that point and their common tangent passes through P since it is the projection of the inflectional tangent to S. The projection of the $n-3$ residual intersections of the cuspidal edge of the tangent cone with S gives $n-3$ cusps of Γ all lying on the common tangent to K and Γ. In the projection of a double contact from P_n each contact goes into a point of K and two consecutive points of Γ so that the projection of the bitangent to S is a bitangent to Γ through P. It is not tangent to K.

Each of the lines through P and a simple basis point of (x) is tangent to K at that simple basis point and has $n-2$ contacts with Γ elsewhere.

For $n=1$ the preceding becomes a three-dimensional construction for the ordinary quadratic transformation.

WELLS COLLEGE,
April, 1921.

DIFFERENTIAL EQUATIONS WITH A CONTINUOUS INFINITUDE OF VARIABLES.

By I. A. Barnett.

The existence of solutions of the simultaneous system of differential equations

$$\frac{du_\rho}{d\tau} = f_\rho(\tau, u_1, \cdots u_\nu), \qquad (\rho = 1 \cdots \nu)$$

has been proved by means of various devices. Moreover the nature of the solutions when regarded as functions of the initial constants has also been treated.*

Some of these results have been extended to differential equations with a denumerable infinitude of variables by Von Koch, Moulton and Hart.†

In 1911, G. Kowalewski discussed the existence theorem for differential equations which involved Schmidt integral power series,‡ and in 1914, Volterra stated an existence theorem for general differential equations involving functions of lines.§ He gives, however, few details of the proof.

It is intended in the first section of this paper to restate this theorem of Volterra, and to give the details of the proof. The problem may be stated somewhat more explicitly as follows. Given the equation

$$(1) \qquad \frac{\partial u(\xi, \tau)}{\partial \tau} = f[\xi, \tau, u(\xi')],$$

where ξ, ξ' are real variables on the range $(0, 1)$, τ is a real variable on $|\tau - \tau_0| \leqq \alpha$, $u(\xi')$ has the range of continuous functions for which $\max |u(\xi') - u_0(\xi')| \leqq \beta$, and $f[\xi, \tau, u]$ is a functional eliminating the argument ξ' and yielding for each ξ, τ, u of the above ranges a real number.

* See, for example, the two articles by Bliss: "The Solutions of Differential Equations of the First Order as Functions of the Initial Values," *Annals of Mathematics*, 2d Series, Vol. 6 (1905), p. 49; "Solutions of Differential Equations as Functions of the Constants of Integration," *Bulletin of the American Mathematical Society*, Vol. XXIV (1918), p. 15. For other references, see Encyclopädie der Mathematischen Wissenschaften, II, A4a, p. 195, 200.

† H. Von Koch, "Ofversigt af Konliga Vetenskaps Akademiens Fordhandligar," Vol. 56 (1899), pp. 395–411.

F. R. Moulton, *Proceedings of the National Academy of Sciences*, Vol. 1, pp. 350–354.

W. L. Hart, "Differential Equations and Implicit Functions in Infinitely Many Variables, "*Transactions of the American Mathematical Society*, Vol. 18 (1917), pp. 125–160.

‡ K. Kowalewski, "Ueber Funktionraume II," *Wiener Berichte*, Vol. 120 (1911), ab. 2a.

§ Volterra, "Equazioni integro-differenziali ed equazioni alle derivate funzionali," *Rendiconti della reale Accademia dei Lincei*, 23 serie V (1914), p. 55.

can be made as small as desired, say less than ϵ. Hence, since the elements (ξ, τ, v_ν) and (ξ, τ, v) are in (A), it follows by (H_2) of Theorem 1 that

$$\left|\int_{\tau_0}^{\tau}\{f[\xi, \tau', v_\nu(\xi', \tau')] - f[\xi, \tau', v(\xi', \tau')]\}d\tau'\right| \leqq \kappa\epsilon|\tau - \tau_0|$$

or

$$\lim_{\nu\to\infty} \int_{\tau_0}^{\tau} f[\xi, \tau', v_\nu(\xi', \tau')]d\tau' = \int_{\tau_0}^{\tau} f[\xi, \tau', v(\xi', \tau')]d\tau'.$$

But, by Lemma 4, $v(\xi, \tau) = \lim_{\nu\to\infty} v_\nu(\xi, \tau)$ so that it follows from the relations (2) that

$$v(\xi, \tau) = u_0(\xi) + \int_{\tau_0}^{\tau} f[\xi, \tau', v(\xi', \tau')]d\tau!$$

as desired.

On differentiating the last relation with respect to τ, one sees that $v_\tau(\xi, \tau)$ is continuous in (B) and satisfies the differential equation (1). Moreover by putting $\tau = \tau_0$ in this same equation, one finds that the solution reduces to the specified initial function $u_0(\xi)$.

It remains only to prove the uniqueness of the solution.

LEMMA 6. *The equation* (1) *has only one solution* $v(\xi, \tau)$ *which is continuous in* (B), *for which* $v_\tau(\xi, \tau)$ *is continuous in* (B) *and which reduces to* $u_0(\xi)$ for $\tau = \tau_0$.

Suppose that besides $v(\xi, \tau)$ there could be another solution $\bar{v}(\xi, \tau)$ having the properties stated in the lemma. Then this solution also would have to satisfy a relation like that in Lemma 5. It would then follow by (H_2) of Theorem 1, since both (ξ, τ, v) and $(\xi, \tau, \bar{v})$ are in (A), that

$$(4) \quad \begin{aligned} |v(\xi, \tau) - \bar{v}(\xi, \tau)| = \left|\int_{\tau_0}^{\tau} \{f[\xi, \tau', v(\xi', \tau')] - f[\xi, \tau', \bar{v}(\xi', \tau')]\}d\tau'\right| \\ \leqq \tau\kappa\beta|\tau - \tau_0|. \end{aligned}$$

If now (H_2) is sequentially applied to the second member of (4) and use is made of the last inequality, one obtains finally the inequality

$$|v(\xi, \tau) - \bar{v}(\xi, \tau)| \leqq \frac{2\beta\kappa^\nu|\tau - \tau_0|^\nu}{\nu!}$$

from which follows the statement of the Lemma by a passage to the limit.

This completes the proof of Theorem I.

The following theorem, which is really a Corollary of Theorem I, will be found useful.

THEOREM II. *Let a set* (A') *be defined by*

$$(A') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \alpha, \qquad ||u - u_0|| \leqq \infty.$$

If the hypotheses of Theorem I are satisfied in this set and if $||f[\xi, \tau, u_0]||$, *then the conclusions of Theorem I will hold for the set of points* (B'),

$$(B') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \alpha.$$

Part (2) of Lemma 3 which deals with proving that the element (ξ, τ, v) is in (A') is of course unnecessary here. The smaller of α and β/γ should in this case be replaced evidently by the number α. The convergence and uniqueness proofs go through precisely as before.

§ 2. SOME DEFINITIONS AND LEMMAS CONCERNING FUNCTIONALS AND DIFFERENCE FUNCTIONS.

Before proceeding to a discussion of other properties of the solutions of equation (I), it will be convenient to give a few definitions and lemmas to which frequent reference will be made in the sequel.

Let η stand for the set of elements $(\eta_1 \cdots \eta_r)$ where the $\eta_1 \cdots \eta_r$ are real variables with ranges which are composed of continuous intervals or discrete sets. For example, η_1 might have the range $1, 1/2, 1/3, \cdots$ and all the other η's vary over $(0, 1)$. Moreover, let u stand for the set of elements $(u_1 \cdots u_s)$ where $u_1 \cdots u_s$ are real continuous functions of ξ' on the interval $0 \leqq \xi' \leqq 1$. Consider now a real-valued functional operation $v[\xi, \eta, u] = v[\xi, \eta_1 \cdots \eta_r, u_1 \cdots u_s)$ defined by the relations

$$(R) \qquad 0 \leqq \xi \leqq 1, \qquad |\eta_i - \eta_i^{(0)}| \leqq \lambda_i \qquad (i = 1 \cdots r),$$
$$||u_i - u_i^{(0)}|| \leqq \mu_i \qquad (i = 1 \cdots s).$$

In other words, $v[\xi, \eta, u]$ is of such a character that whenever the element (ξ, η, u) of the set (R) is given, the correspondence v determines a real number. Unless otherwise specified, attention will be confined to η's which represent sets of variables having continuous ranges.

Definition.—*$v[\xi, \eta, u]$ is said to be continuous in the set (R) if for every positive number ϵ and for every u for which (ξ, η, u) is in (R), there exists another positive number $\delta_{\epsilon, u}$ which is independent of (ξ, η) such that the inequality $|v[\bar{\xi}, \bar{\eta}, \bar{u}] - v[\xi, \eta, u]| \leqq \epsilon$ holds if (ξ, η, u) and $(\xi, \bar{\eta}, \bar{u})$ are in (R) and if $|\bar{\xi} - \xi| \leqq \delta_{\epsilon, u}$, $|\bar{\eta} - \eta| \leqq \delta_{\epsilon, u}$, $||\bar{u} - u|| \leqq \delta_{\epsilon, u}$.*

The notation $|\bar{\eta} - \eta|$ means of course $|\bar{\eta}_i - \eta_i|$, $(i = 1 \cdots r)$ and $||\bar{u} - u||$ stands for $||\bar{u}_i - u_i||$, $(i = 1 \cdots s)$.

LEMMA 1. *If*

(1) *$v_\nu[\xi, \eta, u]$ is continuous in (R),*

(2) *The sequence $\{v_\nu[\xi, \eta, u]\}$ tends to a limit $v[\xi, \eta, u]$ uniformly with respect to all elements $[\xi, \eta, u]$ of (R), then, $v[\xi, \eta, u]$ is continuous in (R).*

In the inequality

$$|v[\bar{\xi}, \bar{\eta}, \bar{u}] - v[\xi, \eta, u]| \leqq |v[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}]|$$
$$+ |v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\xi, \eta, u]| + |v_{\nu_0}[\xi, \eta, u] - v[\xi, \eta, u]|$$

one can choose ν_0 so large that

$$|v[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}]| + |v_{\nu_0}[\xi, \eta, u] - v[\xi, \eta, u]| \leqq \frac{2\epsilon}{3}$$

by the uniformity of the convergence. Moreover by the continuity of $v_{\nu_0}[\xi, \eta, u]$ there exists a number $\delta_{\epsilon, u, \nu_0}$ for which

$$|v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\xi, \eta, u]| \leqq \epsilon/3.$$

Take this $\delta_{\epsilon, u, \nu_0}$ as the δ associated with the continuity of $v[\xi, \eta, u]$.

Let $(\tilde{R})$ be a set of elements $[\tilde{\eta}, \tilde{u}]$ where $\tilde{\eta}$ stands for a set $(\tilde{\eta}_1 \cdots \tilde{\eta}_r)$ with each $\tilde{\eta}_i$ ranging over the whole linear continuum and $\tilde{u}$ stands for the set of functions $(\tilde{u}_1 \cdots \tilde{u}_s)$ where the $\tilde{u}_i$ are arbitrary continuous functions of ξ' on the interval (0, 1). In all that follows in this section the elements (ξ, η, u) and $(\bar{\xi}, \bar{\eta}, \bar{u})$ shall be understood as belonging to (R) and $(\tilde{\eta}, \tilde{u})$ as belonging to $(\tilde{R})$.

Definition of the function $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$.

(1) *a is continuous in $[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ in the sense defined above.*

(2) $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] = \gamma_1 a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2]$, *where* $(\tilde{\eta}_1, \tilde{u}_1)$, $(\tilde{\eta}_2, \tilde{u}_2)$ *are elements of* $(\tilde{R})$ *and* γ_1, γ_2 *are arbitrary real numbers.*

(3) *There exists a positive number μ independent of $[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ for which*

$$|a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]| \leqq \mu \| \tilde{\eta}, \tilde{u} \|.$$

The notation $\| \tilde{\eta}, \tilde{u} \|$ means the larger of $[\| \tilde{\eta} \|, \| \tilde{u} \|]$.

Definition of a difference function.

$a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ is said to be a difference function of $f[\xi, \eta, u]$ if a has the properties (1), (2), (3) *just defined and satisfies the relation*

$$f[\xi, \bar{\eta}, \bar{u}] - f[\xi, \eta, u] = a[\xi, \eta, u, \bar{\eta}, \bar{u}; \bar{\eta} - \eta, \bar{u} - u].$$

Besides the properties (1), (2), (3) of a there will be occasion to consider a fourth property.

(1′) *If in a, one substitutes for u, $\bar{u}$, $\tilde{u}$ the functionals $v'[\xi, \eta', u']$, $v''[\xi, \eta'', u'']$, $v'''[\xi, \eta''', u''']$ respectively, where v', v'', v''' are continuous in the sense above defined, then the resulting expression considered as depending on $\xi, \eta, \bar{\eta}, \tilde{\eta}; \eta', \eta'', \eta''', u', u'', u'''$ is also continuous in the same sense.*

It should be understood in (1′) that η', η'', η''' may denote sets of variables which have both discrete and continuous ranges. If for example η_1 has the range 1, 1/2, 1/3 $\cdots$, then $v(\eta_1)$ is continuous at the point 0 if $|v(\eta_1') - v(0)| \leqq \epsilon$ whenever $|\eta_1'| \leqq \delta$ or what amounts to the same thing $\lim v(\eta_1) = v(0)$ when $\lim \eta_1 = 0$. The u', u'', u''' must have ranges given by (R). This last property is denoted by (1′) because it is similar to (1) but much stronger. It will be understood therefore that whenever (1′) is used it will replace (1).

The following lemmas will be found very useful.

LEMMA 2. *If $f[\xi, \eta, u]$ has a difference function $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ in*

the set (R), *it satisfies the Lipschitz condition described in hypothesis* (H_2) *of Theorem I in this set.*

For, taking $\bar{\eta} = \eta$ and u, $\bar{u}$ as single real continuous functions, one sees from the definition of a difference function and property (3) that

$$|f[\xi, \eta, \bar{u}] - f[\xi, \eta, u]| \leqq \mu \| \bar{u} - u \|.$$

LEMMA 3. *If* $f[\xi, \eta, u]$ *has a difference function in* (R), *it is continuous in* (R) *in the sense described above.*

For, if in the inequality

$|f[\bar{\xi}, \bar{\eta}, \bar{u}] - f[\xi, \eta, u]| \leqq |f[\bar{\xi}, \bar{\eta}, \bar{u}] - f[\bar{\xi}, \eta, u]| + |f[\bar{\xi}, \eta, u] - f[\xi, \eta, u]|$

one takes $\|\bar{\eta} - \eta\|$, $\|\bar{u} - u\|$ sufficiently small, one can make $|f[\xi, \bar{\eta}, \bar{u}] - f[\xi, \eta, u]|$ as small as desired, since it is less than $\mu \|\bar{\eta} - \eta, \bar{u} - u\|$. Furthermore by property (1) of the difference function the second expression on the right tends to zero with $|\bar{\xi} - \xi|$. It is to be noted here that because of the character of the μ the continuity is surely of the type defined in the beginning of this section and even stronger.

LEMMA 4. *Let* $v_\nu[\xi, \eta, u]$ *be a sequence of functionals with the properties:*

(1) *The sequence converges to a limit function* $v[\xi, \eta, u]$.

(2) *Each* $v_\nu[\xi, \eta, u]$ *has a difference function* $b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ *with all of which are associated the same constant* μ.

(3) *The sequence* $\{b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]\}$ *converges to a limit* $b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ *uniformly with respect to all* (ξ, η, u) *in* (R), $(\bar{\xi}, \bar{\eta}, \bar{u})$ *in* (R) *and* $(\tilde{\eta}, \tilde{u})$ *for which* $\|\tilde{\eta}, \tilde{u}\| \leqq 1$.

Then, the limit $b[\xi, \eta, u, \eta, \bar{u}; \tilde{\eta}, \tilde{u}]$ *has the properties* (1), (2) *and* (3) *and is a difference function for* $v[\xi, \eta, u]$.

From (2) one has the relation

$v_\nu[\xi, \bar{\eta}, \bar{u}] - v_\nu[\xi, \eta, u] = b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \eta - \eta, \bar{u} - u], \quad \nu = 1, 2, \cdots,$

so that passing to the limit as is allowable by (1) and (3), it is clear that

$$v[\xi, \bar{\eta}, \bar{u}] - v[\xi, \eta, u] = b[\xi, \eta, u, \bar{\eta}, \bar{u}, \bar{\eta} - \eta, \bar{u} - u].$$

It remains to show that $b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ has the properties (1), (2), (3) of a difference function. By (2) and (3) one sees that

$b[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2]$

$$= \lim_{\nu \to \infty} b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2]$$
$$= \lim_{\nu \to \infty} \{\gamma_1 b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2].$$

Since each term on the right-hand side separately converges, it follows that

$b[\xi, \eta, u, \eta, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2]$

$$= \gamma_1 b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2]$$

which proves property (2). Moreover since each b_ν is continuous in ξ, η, u, $\bar{\eta}$, $\bar{u}$, $\tilde{\eta}$, $\tilde{u}$ and since the sequence $\{b_\nu\}$ converges uniformly to b, it follows from Lemma I that the limit functional b will also be continuous in the same arguments. Finally, property (3) is an immediate consequence of hypothesis (2).

§ 3. CONTINUITY OF THE SOLUTIONS WITH RESPECT TO THE INITIAL ELEMENTS.

Consider now a set (A_0) of elements (ξ, τ, u) defined by the inequalities

$$(A_0) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_{00}| \leqq \alpha + \delta, \qquad \| u - u_{00} \| \leqq \beta + \delta,$$

where (ξ, τ_{00}, u_{00}) is a particular element. Assume that all the hypotheses of Theorem I are satisfied in this set. Consider also a set of elements (ξ, τ, τ_0, u_0) given by the relations,

$$(B_0) \quad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \rho, \qquad |\tau_0 - \tau_{00}| \leqq \delta, \qquad \| u_0 - u_{00} \| \leqq \delta,$$

where ρ as before is the smaller of α and β/γ.

The set (A) associated with every element (τ_0, u_0) for which $|\tau_0 - \tau_{00}| \leqq \delta$ and $\| u_0 - u_{00} \| \leqq \delta$ is a part of (A_0) as is clear from the inequalities,

$$|\tau - \tau_{00}| \leqq |\tau - \tau_0| + |\tau_0 - \tau_{00}| \leqq \alpha + \delta,$$
$$\| u - u_{00} \| \leqq \| u - u_0 \| + \| u_0 - u_{00} \| \leqq \beta + \delta.$$

From this it follows that if one defines a sequence of approximating functionals $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ by equations (2) where now $[\xi, \tau, \tau_0, u_0]$ are all thought of as varying in the set (B_0), one can prove precisely as in Lemma 3, section 1, that these functionals are all defined in (B_0) and that the sequence converges uniformly in (B_0) to a solution $v[\xi, \tau, \tau_0, u_0]$ of the differential equation (1). One has thus secured a set of solutions which for all possible variations of the initial element $[\xi, \tau, \tau_0, u_0]$ in (B) has a common interval of definition $|\tau - \tau_0| \leqq \rho$.

THEOREM III. *If the hypotheses of Theorem I are satisfied in the set (A_0), then in the set (B_0) the solutions $v[\xi, \tau, \tau_0, u_0]$ are continuous functionals of their arguments in the sense of section 2.*

In the notations of section 2 the conclusion of this theorem has reference to a set $(R) = (B_0)$ with the elements $\xi = \xi$, $\eta = (\tau, \tau_0)$ and $u = u_0$. The proof of the theorem can be made to rest on the following

LEMMA. *If*

(1) *$f[\xi, \tau, u]$ satisfies the hypotheses of Theorem I in the set (A_0),*

(2) *$v[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0),*

then, the functionals $g[\xi, \tau, \tau_0, u_0] = f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]$ and $\int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0] d\tau'$ are also continuous in (B_0) in the same sense.

Consider any two elements $[\xi, \tau, \tau_0, u_0]$, $[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ of (B_0). One may write

$$(5)\quad \begin{aligned} |g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0]| \leqq & |g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0]| \\ & + |g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0] - g[\xi, \tau, \tau_0, u_0]|. \end{aligned}$$

Since f satisfies the Lipschitz condition in (A_0), it follows that the first expression on the right will not exceed $\kappa|v[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0]|$. From the continuity of v it follows that this may be made as small as desired if $(\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0)$ is taken sufficiently near (ξ, τ, τ_0, u_0). It is to be remarked moreover that the associated δ is independent of ξ, τ, τ_0. Also, the second expression on the right-hand side of (5) can be made as small as one pleases, since it is equal to $|f[\bar{\xi}, \bar{\tau}, v[\xi', \bar{\tau}, \tau_0, u_0]] - f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]|$ and hence for each u_0 is uniformly continuous in ξ, τ, τ_0. This proves the first part of the Lemma.

Now, set

$$h[\xi, \tau, \tau_0, u_0] = \int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]d\tau'.$$

Then, it is clear that

$$\begin{aligned} |h[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0]| \leqq & \left|\int_{\tau}^{\bar{\tau}} g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0]d\tau'\right| \\ & + \left|\int_{\bar{\tau}_0}^{\tau_0} g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0]d\tau'\right| \\ & + \left|\int_{\tau_0}^{\tau} \{g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau', \tau_0, u_0]\}d\tau'\right|. \end{aligned}$$

Hence, applying the first part of the Lemma and Lemma 1 of section 1 to the first two terms on right, and the fact that $g[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0), one sees that

$$|h[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0]| \leqq \gamma(|\bar{\tau} - \tau| + |\bar{\tau}_0 - \tau_0| + \epsilon|\tau - \tau_0|)$$

if $(\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0)$ is sufficiently near to (ξ, τ, τ_0, u_0) which proves that $h[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0). It is important to note that the uniformity of the continuity of $g[\xi, \tau, \tau_0, u_0]$ with respect to τ was needed in this discussion.

Consider now the approximating functionals

$$(2')\quad \begin{aligned} v_0[\xi, \tau, \tau_0, u_0] &= u_0(\xi), \\ v_{\nu+1}[\xi, \tau, \tau_0, u_0] &= u_0(\xi) + \int_{\tau_0}^{\tau} f[\xi, \tau', v_\nu[\xi', \tau', \tau_0, u_0]]d\tau', \end{aligned}$$
$$\nu = 0, 1, 2, \cdots.$$

Suppose $[\xi, \tau, \tau_0, u_0]$ is any element of (B_0). Then for $\nu = 0$ one has

$$|\Delta v_0| = |v_0[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v_0[\xi, \tau, \tau_0, u_0]| = |\bar{u}_0(\bar{\xi}) - u_0(\xi)|$$

so that

$$|\Delta v_0| \leqq |\bar{u}_0(\bar{\xi}) - u_0(\bar{\xi})| + |u_0(\bar{\xi}) - u_0(\xi)|.$$

But, if ϵ is arbitrarily assigned there exists a δ_{ϵ, u_0} independent of ξ such that if $|\bar{\xi} - \xi| \leqq \delta_{\epsilon, u_0}$ then $|u_0(\bar{\xi}) - u_0(\xi)| \leqq \epsilon$. This δ is independent of ξ

since u_0 is a continuous function on a closed interval. Also there exists a δ_ϵ such that if $\| \bar{u}_0 - u_0 \| \leqq \delta_\epsilon$ then $| \bar{u}_0(\bar{\xi}) - u_0(\bar{\xi}) | \leqq \epsilon$. This shows that $v_0[\xi, \tau, \tau_0, u_0]$ possesses precisely that type of continuity heretofore considered and may therefore be used as the $v[\xi, \tau, \tau_0, u_0]$ of the preceding Lemma. Thus by successive applications of this Lemma one gets the result that all the $v_\nu[\xi, \tau, \tau_0, u_0]$ are continuous in (B_0). Furthermore, as has already been remarked at the beginning of this section, the sequence $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ converges to a solution $v[\xi, \tau, \tau_0, u_0]$ uniformly in (B_0). Hence, using Lemma I of section 2 with $\eta = (\tau, \tau_0)$, $u = u_0$ one obtains the conclusion of Theorem III.

§ 4. DIFFERENTIABILITY OF THE SOLUTIONS WITH RESPECT TO THE INITIAL ELEMENTS.

The main result of this section is embodied in the following

THEOREM IV. *If*

(H_1) *The hypotheses of Theorem I are satisfied in* (A_0),

(H_2) $f[\xi, \tau, u]$ *has a difference function* $a[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau}, \bar{u}]$ *in* (A_0),

$$f[\xi, \bar{\tau}, \bar{u}] - f[\xi, \tau, u] = a[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau} - \tau, \bar{u} - u],$$

(H_3) *The difference function of* f *has the additional property* (1′) *of section 2 in* (A_0),

then, the solution $v[\xi, \tau, \tau_0, u_0]$ *has in* (B_0) *a difference function* $b[\xi, \tau, \tau_0, u_0\bar{\tau}, \bar{\tau}_0\bar{u}_0; \bar{\tau}\bar{\tau}_0\bar{u}_0]$.

The style of proof will be to interpret the sequence $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ with $\eta = (\tau, \tau_0)$ and $u = u_0$ of Lemma 4, section 2, as the sequence of approximating functionals (2′). If then one can show that all the hypotheses of that Lemma are satisfied here, Theorem IV will be proved.

LEMMA 1. *If*

(1) $f[\xi, \tau, u]$ *satisfies the hypotheses of Theorem IV*,

(2) $v[\xi, \tau, \tau_0, u_0]$ *has a difference function* $b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0\bar{u}_0; \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ *in* (B_0),

(3) $(\xi, \tau, v[\xi', \tau, \tau_0, u_0])$ *is in* (A_0) *for every* (ξ, τ, τ_0, u_0) *of* (B_0),

then, $g[\xi, \tau, \tau_0, u_0] = f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]$ *and* $\int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]d\tau'$ *also have difference functions in* (B_0).

By hypothesis (1)

$$g[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0] = a[\xi, \tau, v, \bar{\tau}, \bar{v}; \bar{\tau} - \tau, \bar{v} - v],$$

where $v = v[\xi, \tau, \tau_0, u_0]$ and $\bar{v} = v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$. Moreover

$$v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v[\xi, \tau, \tau_0, u_0] \\ = b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0].$$

Hence,

$$g[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0] = c[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0],$$

where

$$c[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0\bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0] = a[\xi, \tau, v, \bar{\tau}, \bar{v}; \tilde{\tau}, b[\xi', \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0]].$$

It remains to show that c has properties (1), (2) and (3) of a difference function. Property (1) of c is an immediate consequence of the property (1′) possessed by the difference function a. Property (2) follows from property (2) of b and a. Since (ξ, τ, v), $(\xi, \bar{\tau}, \bar{v})$ are in (A_0), it is clear from property (3) of a that $|c| \leqq \mu \| \tilde{\tau}, b \|$ and hence from property (3) of b that $|c| \leqq \mu\lambda \| \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0 \|$, where λ is the constant associated with b. Thus property (3) of c is proved. This completes the proof that $g[\xi, \tau, \tau_0, u_0]$ has a difference function in (B).

Consider now the functional $h[\xi, \tau, \tau_0, u_0] = \int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]$. By means of an easy computation it is found that

$$\begin{aligned} k[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0] &= \tilde{\tau}\int_0^1 g[\xi, \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0]d\theta \\ &+ \tilde{\tau}_0\int_1^0 g[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0]d\theta \\ &+ \int_{\tau_0}^{\tau} c[\xi, \tau', \tau_0, u_0, \tau', \bar{\tau}_0, \bar{u}_0; 0, \tilde{\tau}_0, \tilde{u}_0]d\tau', \end{aligned} \tag{6}$$

where

$k[\xi, \tau, \tau_0, u_0\bar{\tau}\bar{\tau}_0\bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0] = h[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0]$. Properties (2) and (3) of k follow readily from the identity (6), properties (2) and (3) of g, the continuity of c, and elementary properties of integrals. The proof of property (1) follows readily from the fact that g and c are continuous in the sense of section 2.

LEMMA 2. *Every approximating functional $v_\nu[\xi, \tau, \tau_0, u_0]$ has a difference function b_ν in (B_0) if f satisfies the hypotheses of Theorem IV.*

It is clear from (2′) that $v_0[\xi, \tau, \tau_0, u_0]$ has a difference function $b_0 = \tilde{u}_0$. Then from Lemma 1, it will follow that every v_ν has a difference function b_ν where

$$b_\nu[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0] = v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v_\nu[\xi, \tau, \tau_0, u_0], \qquad \nu = 1, 2, \cdots.$$

It can be shown from the defining equations (2′) that

$$\begin{aligned} b_{\nu+1}[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0] &= \tilde{u}_0(\xi) \\ &+ \tilde{\tau}\int_0^1 f[\xi, \tau + \theta(\bar{\tau} - \tau), v_\nu[\xi', \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0]d\theta \\ &+ \tilde{\tau}_0\int_1^0 f[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), v_\nu[\xi', \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0]d\theta \\ &+ \int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, b_\nu[\xi', \tau', \tau_0, u_0, \tau'\bar{\tau}_0, \bar{u}_0; 0, \tilde{\tau}_0, \tilde{u}_0]d\tau' \end{aligned} \tag{7}$$

where v_ν and $\bar{v}_\nu$ stand for $v_\nu[\xi, \tau, \tau_0, u_0]$ and $v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ respectively.

LEMMA 3. *The difference functions b_ν of the approximating functionals have a common constant λ associated with them.*

In the first place it is clear from the definition of b_0 that

$$|b_0| \leqq |||\bar{\tau}|, |\bar{\tau}_0|, |\bar{u}_0|||.$$

Furthermore it follows from (7) and the properties of f and a that if there exist functions $b_\nu(\tau)$ such that

$$|b_\nu| \leqq \beta_\nu(\tau)\{|||\bar{\tau}|, |\bar{\tau}_0|, |\bar{u}_0|||\},$$

then

$$|b_{\nu+1}| \leqq \{1 + 2\gamma + |\int_{\tau_0}^{\tau} \kappa\beta_\nu(\tau')d\tau'|\}\{|||\bar{\tau}|, |\bar{\tau}_0|, |\bar{u}_0|||\},$$

so that

$$\beta_{\nu+1}(\tau) = 1 + 2\gamma + |\int_{\tau_0}^{\tau} \kappa\beta_\nu(\tau')d\tau'|.$$

One obtains therefore the following expressions of the functions $\beta_\nu(\tau)$

$$\beta_0 = 1,$$

$$\beta_1 = (1 + 2\gamma) + \frac{\kappa|\tau - \tau_0|}{1!},$$

$$\beta_2 = (1 + 2\gamma) + \frac{(1 + 2\gamma)^{\kappa|\tau - \tau_0|}}{1!} + \frac{\kappa^2|\tau - \tau_0|^2}{2!} \leqq (1 + 2\gamma)e^{\kappa\rho},$$

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$$\beta_{\nu+1} = (1 + 2\gamma)\left\{1 + \frac{\kappa|\tau - \tau_0|}{1!} + \cdots \frac{\kappa^\nu|\tau - \tau_0|^\nu}{\nu!}\right\} + \frac{\kappa^{\nu+1}|\tau - \tau_0|^{\nu+1}}{(\nu + 1)!} \leqq (1 + 2\gamma)e^{\kappa\rho}.$$

Hence the common constant λ is given by $(1 + 2\gamma)e^{\kappa\rho}$ where $\rho = |\tau - \tau_0|$.

LEMMA 4. *If $p_\nu[\xi, \tau, \tau_0, u_0]$ and $p[\xi, \tau, \tau_0, u_0]$ are continuous in (B_0) and $\lim p_\nu = p$ uniformly in (B_0), then $\lim c_\nu(p_\nu) = c(p)$, where*

$$c_\nu[p_\nu] = \int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p_\nu[\xi, \tau, \tau_0, u_0]]d\tau',$$
$$c[p] = \int_{\tau_0}^{\tau} a[\xi, \tau', v, \tau', \bar{v}; 0, p[\xi, \tau, \tau_0, u_0]]d\tau'.$$

It is seen that $c[p]$ is obtained from $c_\nu[p_\nu]$ by substituting for v_ν and $\bar{v}_\nu$ the limits v and $\bar{v}$ respectively as $\nu \to \infty$, and putting for p_ν the limit p. One has from the linearity property of

$$c[p] - c_\nu[p_\nu] = \int_{\tau_0}^{\tau} \{a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p] - a[\xi, \tau', v, \tau', \bar{v}; 0, p]\}d\tau' + \int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p_\nu - p].$$

The integrand of the second term on the right does not exceed $\mu \,||\, p_\nu - p \,||$ so that it can be made as small as desired because of the uniformity of the

approach of p_ν to p. In the first term regard v_ν and $\bar{v}_\nu$ as depending upon $\nu', \xi, \tau, \tau_0, u_0$ and $\nu', \xi, \bar{\tau}, \bar{\tau}_0, u_0$, where $\nu' = 1/\nu$. Since $\lim_{\nu\to\infty} v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] = v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ and $\lim_{\nu\to\infty} v_\nu[\xi, \tau, \tau_0, u_0] = v[\xi, \tau, \tau_0, u_0]$ uniformly in (B_0), it follows that $v[\nu', \xi, \tau, \tau_0, u_0]$ and $v[\nu', \xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ are continuous functions of their arguments in the sense of section 2 so that property (1′) of a difference function is applicable, thus showing that $\lim_{\nu\to\infty} c_\nu[p_\nu] = c[p]$ as desired.

LEMMA 5. *The sequence* $\{b_\nu\}$ *converges uniformly in* (B_0) *to a limit* b *given by the expression* (11) *below.*

By Lemma 2 of this section one may write

$$(8) \qquad b_{\nu+1} = d_\nu + c_\nu[b_\nu],$$

where

$$(9) \quad d_\nu = \tilde{u}_0(\xi) + \tilde{\tau}\int_0^1 f[\xi, \tau + \theta(\bar{\tau} - \tau), v_\nu[\xi', \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0]]d\theta;$$
$$+ \tilde{\tau}_0 \int_1^0 f[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), v_\nu[\xi', \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0]]d\theta$$

and

$$(10) \quad c_\nu[b_\nu] = \int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, b_\nu[\xi', \tau', \tau_0, u_0, \tau', \bar{\tau}_0\bar{u}_0; 0, \tilde{\tau}_0, \tilde{u}_0]]d\tau'.$$

Repeating (8) ν times one obtains

$$b_{\nu+1} = d_\nu + c_\nu[d_{\nu-1}] + c_\nu c_{\nu-1}[d_{\nu-2}] + \cdots + c_\nu c_{\nu-1}\cdots c_1[b_0],$$

where the notation explains itself.

Consider also the infinite sum

$$(11) \qquad b = d + c[d] + c^2[d] + \cdots + c^\kappa[d] + \cdots,$$

where d and $c[d]$ have precisely the form of d_ν and $c_\nu[d_\nu]$ only with v, $\bar{v}$ substituted for v_ν and $\bar{v}_\nu$ respectively, and where c^κ means that the operation c is repeated κ times. It has already been seen that each term of (10) is dominated by the corresponding term of $(1 + 2\gamma)e^{\kappa\rho}\,|||\tilde{\tau}|, |\tilde{\tau}_0|, |u_0|||$. In a similar way it can be shown that the series for the last expression also dominates term by term the series (11). Hence there exists an integer κ_0 of such a character that all the terms of $b_{\nu+1}$ after the κ_0th have a sum which is in absolute value at most $\epsilon/3$ where ϵ is an arbitrarily assigned positive number. The same is true for the series representing b. Moreover, by repeated application of Lemma 4, one sees that there exists an index ν_0 such that if $\nu > \nu_0$, the first κ_0 terms of $b_{\nu+1}$ can be made to differ from the first κ_0 terms of b by a number which in absolute value is at most $\epsilon/3$. Hence $\lim b_\nu = b$ uniformly in (B_0).

Recalling now that the ν_ν converge to the solution v one sees that the hypothesis $H_{(1)}$ of Lemma 4, section 2 is satisfied. Furthermore, hypothesis $H_{(2)}$ of that Lemma is contained in Lemmas 2 and 3 of this section and $H_{(3)}$ is nothing but Lemma 5. Thus the conclusions of Lemma 4 are applicable

so that the solution $v[\xi, \tau, \tau_0, u_0]$ has a difference function b in (B_0) as desired.

§ 5. DIFFERENTIAL EQUATIONS INVOLVING SCHMIDT INTEGRAL POWER SERIES.

Let $u(\xi)$ be a real continuous function on the range (01). Then the function of $r+1$ arguments

$$u^p(\xi)u^{p_1}(\xi_1)\cdots u^{p_r}(\xi_r) \tag{12}$$

where $\xi_1 \cdots \xi_r$ also range over (01) and $p,\ p_1, \cdots p_r$ are equal or distinct integers is evidently continuous. If now (12) is multiplied by an arbitrary function $\kappa(\xi, \xi_1, \cdots \xi_r)$ continuous in its arguments and the whole expression is integrated with respect to $\xi_1 \cdots \xi_r$, then the resulting function

$$w[\xi, u] = \int_0^1\cdots\int_0^1\kappa(\xi, \xi_1, \cdots \xi_r)u^p(\xi)u^{p_1}(\xi_1)\cdots u^{p_r}(\xi_r)d\xi_1\cdots d\xi_r{}^* \tag{13}$$

is again a continuous function of ξ. The expression

$$\mathfrak{A}[\xi, u] = w_1[\xi, u] + w_2[\xi, u] + \cdots + w_n[\xi, u]$$

is called a homogeneous integral power form of the mth order (Schmidt) if $w_1 \cdots w_n$ all have the form (13), and if for each w_i, $p + p_1 + \cdots + p_r = m$, where we may suppose always $m \leqq p$. For example, an integral power form of the 2d order has the following form,

$$\int_0^1\cdots\int_0^1\{\kappa_{200}u^2(\xi) + \kappa_{110}u(\xi)u(\xi_1) + \kappa_{020}u^2(\xi_1) + \kappa_{011}u(\xi_1)u(\xi_2)\}d\xi_1\cdots d\xi_r,$$

when all the κ's are arbitrary continuous functions of the $r+1$ arguments $\xi, \xi_1, \cdots \xi_r$. This evidently reduces to

$$\alpha(\xi)u^2(\xi) + u(\xi)\int_0^1\beta(\xi, \xi_1)u(\xi_1)d\xi_1 + \int_0^1\gamma(\xi, \xi_1)u^2(\xi_1)d\xi_1 \\ + \int_0^1\int_0^1\epsilon(\xi, \xi_1, \xi_2)u(\xi_1)u(\xi_2)d\xi_1 d\xi_2.$$

The κ's or the $\alpha, \beta, \gamma \cdots$ are called the coefficients by the form.

Let $\mathfrak{A}[\xi, u]$ be an integral power form of the mth order and let u be replaced by $u + \tilde{u}$, where $\tilde{u}$ is a continuous function of ξ. Then the sum of the terms involving the first power of $\tilde{u}$ is called the *first differential of* $\mathfrak{A}[\xi, u]$ and is denoted by $\mathfrak{A}[\xi, u, \tilde{u}]$ (Kowalewski). It is clear that when u is fixed $\mathfrak{A}[\xi, u, \tilde{u}]$ is an integral power form of the 1st order in $\tilde{u}$ and for $\tilde{u}$ fixed, of the $(m-1)$st order in u. This differential is precisely the difference function defined in section 2 when $\bar{u} = u$. For example to compute the difference function of an integral power form of the 2d order one considers the expression

$$\mathfrak{A}[\xi, \bar{u}] - \mathfrak{A}[\xi, u] = \alpha(\xi)[\bar{u}^2(\xi) - u^2(\xi)] + \bar{u}(\xi)\int_0^1\beta(\xi, \xi_1)\bar{u}(\xi_1)d\xi_1 \\ - u(\xi)\int_0^1\beta(\xi, \xi_1)u(\xi_1)d\xi_1 + \int_0^1\gamma(\xi, \xi_1)[\bar{u}^2(\xi_1) - u^2(\xi_1)]d\xi_1 \\ + \int_0^1\int_0^1\epsilon(\xi, \xi_1, \xi_2)[\bar{u}(\xi_1)\bar{u}(\xi_2) - u(\xi_1)u(\xi_2)]d\xi_1 d\xi_2,$$

* It is clear that the variables of integration $\xi_1, \cdots \xi_r$ may be so arranged that $p_1 \geqq p_2 \geqq \cdots \geqq p_r$ and this will be assumed throughout the section. For an account of this work see Kowalewski, loc. cit.

so that the difference function is

$$a[\xi, u, \bar{u}; \tilde{u}] = \alpha(\xi)[\bar{u}(\xi) + u(\xi)]\tilde{u}(\xi) + \tilde{u}(\xi)\int_0^1\beta(\xi, \xi_1)\bar{u}(\xi_1)d\xi_1 + u(\xi)\int_0^1 \beta(\xi, \xi_1)\tilde{u}(\xi_1)d\xi_1 + \int_0^1\gamma(\xi, \xi_1)[\bar{u}(\xi_1) + u(\xi_1)]\tilde{u}(\xi_1)d\xi_1 + \int_0^1\int_0^1\epsilon(\xi, \xi_1, \xi_2)[\tilde{u}(\xi_1)\bar{u}(\xi_2) + u(\xi_1)\tilde{u}(\xi_2)]d\xi_1 d\xi_2.$$

The differential is obtained by putting $\bar{u} = u$ and is

$$\mathfrak{A}[\xi, u, u; \tilde{u}] = 2\alpha(\xi)u(\xi)\tilde{u}(\xi) + \tilde{u}(\xi)\int_0^1\beta(\xi, \xi_1)u(\xi_1)d\xi_1 + u(\xi)\int_0^1\beta(\xi, \xi_1)\tilde{u}(\xi_1)d\xi_1 + 2\int_0^1\gamma(\xi, \xi_1)u(\xi_1)\tilde{u}(\xi_1)d\xi_1 + 2\int_0^1\int_0^1\epsilon(\xi, \xi_1, \xi_2)\tilde{u}(\xi_1)u(\xi_2)d\xi_1 d\xi_2.$$

Consider now the first differential $\mathfrak{A}[\xi, u, \tilde{u}]$ of the integral power form $\mathfrak{A}[\xi, u]$ of the mth order. If one puts in this $u + \hat{u}$ for u and arranges the result according to the powers of $\hat{u}$, then the sum of the terms of the first power in $\hat{u}$ is called the second differential of $\mathfrak{A}[\xi, u]$ and is denoted by $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$. In a similar way one could define the higher differentials of $\mathfrak{A}[\xi, u]$. One may easily deduce the formula

$$(14) \qquad \mathfrak{A}[\xi, u + \tilde{u}] = \mathfrak{A}[\xi, u] + \frac{\mathfrak{A}[\xi, u; \tilde{u}]}{1!} + \frac{\mathfrak{A}[\xi, u; \tilde{u}, \tilde{u}]}{2!} + \cdots$$

with the analogues of Euler's formulas

$$\mathfrak{A}[\xi, u; u] = m\mathfrak{A}[\xi, u], \qquad \mathfrak{A}[\xi, u; u, u] = m(m-1)\mathfrak{A}[\xi, u], \cdots$$

which may be obtained from the obvious relation

$$\mathfrak{A}[\xi, cu] = c^m\mathfrak{A}[\xi, u],$$

where c is an arbitrary constant.

If in $\mathfrak{A}[\xi, u]$ all the coefficients are replaced by their absolute values and $u(\xi)$ by 1, one obtains a function of ξ whose maximum is called the height (Höhe) of $\mathfrak{A}[\xi, u]$. In a similar way one may define the height of $\mathfrak{A}[\xi, u; \tilde{u}]$, $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$, etc., by replacing the coefficients by their absolute values and u, $\tilde{u}$, $\hat{u}$, $\cdots$ by 1 and then taking the maximum with respect to ξ. If μ is the height of $\mathfrak{A}[\xi, \mu]$, then by Euler's relations $m\mu$ is the height of $\mathfrak{A}[\xi, u; \tilde{u}]$, $m(m-1)\mu$ is the height of $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$ etc.

One may now with Schmidt consider infinite sums of integral power forms (integral power series), viz.,

$$\mathfrak{P}[\xi, u] = \mathfrak{A}_0[\xi, u] + \mathfrak{A}_1[\xi, u] + \mathfrak{A}_2[\xi, u] + \cdots,$$

where $\mathfrak{A}_p[\xi, u]$ is an integral power form of the pth order. Let μ_p represent the heights of $\mathfrak{A}_p[\xi, u]$ and consider the series

$$\mu_0 + \mu_1 x + \mu_2 x^2 + \cdots.$$

If the radius of convergence R of this power series is different from zero,

then the integral power series is said to be *regular* for $|u(\xi)| \leqq R$. If $|u(\xi)| \leqq r < R$, then the above power series is dominated by

$$P(r) = \mu_0 + \mu_1 r + \mu_2 r^2 + \cdots$$

since $|\mathfrak{A}_p[\xi, u]| \leqq \mu_p r^p$. If $|u(\xi)| \leqq r$ and $|\bar{u}(\xi)| \leqq s$, $r + s < R$, one has immediately the inequalities

$$(15) \qquad \begin{gathered} |\mathfrak{A}_p[\xi, u]| \leqq \mu_p r^p, \qquad |\mathfrak{A}_p[\xi, u; \bar{u}]| \leqq p\mu_p r^{p-1} s, \\ |\mathfrak{A}_p[\xi, u; \bar{u}, \bar{u}]| \leqq p(p-1)\mu_p r^{p-2} s^2. \end{gathered}$$

Moreover Kowalewski shows by means of these that the following expansion is valid:

$$(16) \quad \mathfrak{P}[\xi, u + \bar{u}] = \mathfrak{P}[\xi, u] + \mathfrak{P}[\xi, u; \bar{u}] + \tfrac{1}{2}\mathfrak{P}[\xi, u; \bar{u}, \bar{u}] + \cdots,$$

where

$$\begin{aligned} \mathfrak{P}[\xi, u; \bar{u}] &= \mathfrak{A}_1[\xi, u; \bar{u}] + \mathfrak{A}_2[\xi, u; \bar{u}] + \cdots, \\ \mathfrak{P}[\xi, u; \bar{u}, \bar{u}] &= \mathfrak{A}_2[\xi, u; \bar{u}, \bar{u}] + \mathfrak{A}_3[\xi, u; \bar{u}, \bar{u}] + \cdots, \text{etc.,} \end{aligned}$$

From the inequalities (15) one obtains directly the dominance relation

$$(17) \quad \sum_{p,q}^{0\infty} \frac{1}{p!} |\mathfrak{A}_{p+q}[\xi, u; \overset{1}{\bar{u}} \overset{2}{\bar{u}} \cdots \overset{p}{\bar{u}}]| \ll \sum_{p,q}^{0\infty} \frac{(p+q)\cdots(q+1)}{p!} \mu_{p+q} r^q s^p,$$

which may also be written with the help of (16) supposing $r + s < R$,

$$(18) \qquad |\mathfrak{P}[\xi, u + \bar{u}]| \ll P(r + s).$$

Furthermore

$$(18') \qquad \begin{aligned} |\mathfrak{P}[\xi, u; \bar{u}]| &\ll P'(r)s, \\ |\mathfrak{P}[\xi, u; \bar{u}, \bar{u}]| &\ll P''(r)s^2, \qquad \text{etc.,} \end{aligned}$$

where $P'(r)$, $P''(r)$, etc., are the first, second, etc., derived series of $P(r)$.

It is desired now to investigate the differential equation containing integral power series

$$(19) \qquad \begin{cases} \dfrac{\partial u(\xi, \tau)}{\partial \tau} = \mathfrak{P}[\xi, u], \\ u(\xi, 0) = 0 \qquad (\xi) \end{cases}$$

and see if the theorems proved in sections 1–4 are applicable. Kowalewski proves that such equations have solutions by a method very much analogous to that employed in proving the existence theorem for differential equations involving analytic functions. He shows in fact that the solution of

$$\frac{\partial u(\xi, \tau)}{\partial \tau} = \mathfrak{P}[\xi, u]$$

may be developed as a power series in τ with regular integral power series in u for coefficients.

It will be shown that the theory of the system (19) may be considered as a special instance of the theory described in this paper.

The following theorem is true.

THEOREM I′. *If* $\mathfrak{P}[\xi, u]$ *is a regular integral power series for* $\| u \| \leqq R$, *then the system* (19) *has a unique solution* $v(\xi, \tau)$ *continuous in*

$$(B') \qquad 0 \leqq \xi \leqq 1, \quad |\tau| \leqq \rho, \qquad \left(\rho = \frac{\beta}{P(R)}, \quad \beta = \frac{R}{3}\right).$$

The set (A') is now given by

$$(A') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau| < \infty, \qquad \| u \| \leqq \beta.$$

In the first place, since each term of the integral power series is a continuous function of ξ, and since the series is uniformly convergent with regard to ξ for each fixed u of (A'), it is clear that hypothesis (H_1) of Theorem I is satisfied. To prove (H_2), replace in (16) the variables $u, \bar{u}$ by $u, u' - u$ respectively; with u, u' both satisfying (A'). Then, for each such u and u', one has

$$(20) \quad \mathfrak{P}[\xi, u'] - \mathfrak{P}[\xi, u] = \mathfrak{P}[\xi, u; u' - u] + \tfrac{1}{2}\mathfrak{P}[\xi, u; u' - u, u' - u] + \cdots$$

so that by the inequalities (18′), one finds that the right-hand side of the last equation is dominated by

$$s\left[P'(\beta) + \frac{s}{2}P''(\beta) + \cdots\right]$$

where $s = \| u' - u \|$. Hence, since $s = \| u' - u \| \leqq 2\beta$ and $\gamma + s \leqq 3\beta < \kappa$, it follows from (17) and (18) that

$$|\mathfrak{P}[\xi, u'] - \mathfrak{P}[\xi, u]| \leqq sP'(\beta + s) \leqq P'(R) \| u' - u \|.$$

Thus the hypotheses of Theorem I are satisfied. From the statements just preceding equation (15), it is clear that the value $\gamma = P(R)$ is an upper bound for $\mathfrak{P}[\xi, u]$.

The theorem on the continuity with regard to the initial elements reads precisely as Theorem III and will not be stated here.

THEOREM IV′. *If* $\mathfrak{P}[\xi, u]$ *is a regular integral power series for* $\| u \| \leqq R$, *then in the set*

$$(B_0') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \rho, \qquad |\tau_0| \leqq \delta, \qquad \| u_0 \| \leqq \delta,$$

the solution $v[\xi, \tau, \tau_0, u_0]$ *has a difference function* $b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau}, \bar{\tau}_0 \bar{u}_0]$

The set corresponding to (A_0) is here given by

$$(A_0') \quad 0 \leqq \xi \leqq 1, \qquad -\infty < \tau < \infty, \qquad \| u \| \leqq \beta + \delta, \qquad (\beta + \delta < R/3).$$

It is requisite to show that the hypotheses of Theorem IV are satisfied. The proof that the hypotheses of Theorem I are satisfied in (A_0') is the same as that given in Theorem I′. To show that $\mathfrak{P}[\xi, u]$ has a difference function one makes use of the expansion (20). Consider the right-hand side with the last argument $u' - u$ of each term replaced by $\bar{u}$,

$$a[\xi, u, u'; \bar{u}] = \mathfrak{P}[\xi, u; \bar{u}] + \tfrac{1}{2}\mathfrak{P}[\xi, u; u' - u, \bar{u}] + \cdots.$$

The linearity of the functional a in $\bar{u}$ is clear from the definitions of $\mathfrak{P}[\xi, u; \bar{u}]$, $\mathfrak{P}[\xi, u; u' - u, \bar{u}]$, etc. Furthermore, the functional a is continuous for every continuous function $\bar{u}$. For, set $s = ||\, u' - u\,||$ and $S = ||\bar{u}||$; then, by inequalities analogous to (15), (17) and (18′), it follows that

$$|\mathfrak{P}[\xi, u; \bar{u}]| \ll P'(\beta)S \ll P'(R)S,$$
$$|\mathfrak{P}[\xi, u; u' - u, \bar{u}]| \ll P''(\beta)Ss \ll P''(R)Ss, \text{ etc.},$$

Hence, the series for $a[\xi, u, u'; \bar{u}]$ is dominated by $P'(\beta + s)S$ or $P'(R)S$; i.e., for a fixed $\bar{u}$ it converges uniformly with respect to its other arguments and consequently represents a continuous functional for each fixed $\bar{u}$. The constant μ associated with the difference function is $P'(R)$.

It remains to prove that a has the property (1′). In the first place it can readily be shown by a method of proof similar to that used in the proof of Lemma 1, section 2, that, if each of a sequence of functionals has the property (1′), and the sequence converges uniformly in the set under consideration, then the limit functional also has the property (1′). Since the above series for a does indeed converge uniformly, it suffices to prove that the difference function of (13) has the property (1′). But it follows from the definition of a difference function and from the form of (13) that the difference function of (13) is a sum of integrals of the same type except that in the integrand product there occur two functions u, $\bar{u}$ instead of a single function u. That integrals of this last type have property (1′), follows very readily from the fact that the integrals are continuous functionals of u and $\bar{u}$.

THE FACTORIZATION OF THE RATIONAL PRIMES IN A CUBIC DOMAIN.

By G. E. Wahlin.

I. Introduction.

The object of this paper is to determine a set of functions of the coefficients of a cubic equation whose character, with respect to the prime modulus p, completely determines the factorization of p in the cubic domain defined by a root of the equation.

Since every cubic equation can, by a linear transformation, be transformed into the form $x^3 + Cx + D = 0$, where C and D are rational integers, there is no loss in assuming the cubic equation in the following discussion to be of this form. Moreover if $C = p^\lambda \cdot C_1$ and $D = p^\mu \cdot D_1$ where C_1 and D_1 are prime to p, when $\lambda \geqq 2$ and $\mu \geqq 3$ the roots of the equation may be divided by a power of p and the equation thus further reduced. We shall therefore throughout the following pages suppose that $0 \leqq \lambda < 2$ or $0 \leqq \mu < 3$.

We shall denote $x^3 + Cx + D$ by $F_3(x)$ and $3Cx^2 + 9Dx - C^2$ by $F_2(x)$. The discriminant of $F_3(x) = 0$ is $-27D^2 - 4C^3$ and shall be denoted by Δ_3. The discriminant of $F_2(x) = 0$ is $-3\Delta_3$ and shall be denoted by Δ_2.

Let μ_1 and μ_2 be the roots of

$$(1) \qquad F_2(x) = 0.$$

Since $F_3(x)$ is supposed to be irreducible $\Delta_3 \neq 0$ and hence $\Delta_2 \neq 0$ and $\mu_1 \neq \mu_2$.

If $\mu_1 = (-9D - \sqrt{\Delta_2})/6C$ and $\mu_2 = (-9D + \sqrt{\Delta_2})/6C$, we shall write $(6C\mu_2)^n = \psi_n(-9D, \Delta_2) + \varphi_n(-9D, \Delta_2)\sqrt{\Delta_2}$ where ψ_n and φ_n are polynomials. Hence $(6C\mu_2)^n - (6C\mu_1)^n = 2\varphi_n(-9D, \Delta_2)\sqrt{\Delta_2}$.

We shall next apply, to the cubic $F_3(x) = 0$, the non-singular transformation

$$x = \frac{\mu_1 y + \mu_2}{y + 1}.$$

After simplifying the new equation has the form $F_3(\mu_1)y^3 + F_3(\mu_2) = 0$. Since

$$(2) \qquad F_3(x) = F_2(x)\left(x - \frac{3D}{C}\right) \cdot \frac{1}{3C} - \frac{\Delta_3}{3C^2}x$$

and μ_1 and μ_2 are roots of (1), we have $F_3(\mu_i) = -(\Delta_3/3C^2)\mu_i$ $(i = 1, 2)$

and hence the transformed cubic may be written in the form

$$y^3 + \frac{\mu_2}{\mu_1} = 0 \tag{3}$$

and the roots of this are $\rho^i \sqrt[3]{-(\mu_2/\mu_1)}$ $(i = 0, 1, 2)$, where ρ is a primitive cube root of unity. The roots of $F_3(x) = 0$ are therefore

$$\alpha_i = \frac{\mu_1 \rho^i \sqrt[3]{-\frac{\mu_2}{\mu_1}} + \mu_2}{\rho^i \sqrt[3]{-\frac{\mu_2}{\mu_1}} + 1} \quad (i = 0, 1, 2),$$

and in simplified form $\alpha_0 = -\mu_1^{1/3} \cdot \mu_2^{1/3} (\mu_1^{1/3} + \mu_2^{1/3})$.

The above implies that $C \neq 0$, but this does not invalidate the applications which shall be made of it, because it will be seen that in all cases where it is used C is necessarily different from zero.

The theory underlying the following development is that of the application of the p-adic numbers to the study of the algebraic numbers.*

II. Eisensteinian Functions.

A polynomial of the form

$$E(x) = x^n + p^{e_1} a_1 x^{n-1} + p^{e_2} a_2 x^{n-2} + \cdots + p^{e_{n-1}} a_{n-1} x + p^{e_n} a_n,$$

in which all the coefficients except that of the highest power of x are divisible by the prime p is called an Eisensteinian function. Eisenstein's theorem, that every such polynomial in which $e_n = 1$ is irreducible, is well known. O. Perron† has generalized this theorem as follows. The algebraic equation

$$x^n + p^{[e/n]+1} a_1 x^{n-1} + p^{[2e/n]+1} a_2 x^{n-2} + \cdots + p^{[(n-1)e/n]+1} a_{n-1} x + p^e a_n = 0,$$

where $a_1, a_2, \cdots a_n$ are arbitrary integers and a_n prime to p and e prime to n is irreducible. The proof of this theorem consists in showing that, in the domain defined by a root of the given equation, p is the nth power of a prime ideal and hence the domain must be of degree n. We note here that this method is also sufficient to show the irreducibility in $k(p)$.

In the "Theorie der Algebraischen Zahlen" Hensel shows that every factor of an Eisensteinian function in $k(p)$ is an Eisensteinian function and hence the number of factors cannot exceed the exponent of p in the last term.

From Perron's theorem we can conclude the following fact:

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* Hensel, "Theorie der Algebraischen Zahlen." Author, *Transactions Am. Math. Soc.*, Vol. 16. A new development of the theory for quadratic domains was published by Hensel in *Crelle's Journal*, Vol. 144. In the author's paper, here referred to, he gives an extension of this to the general case. Of importance, in the following pages, is the isomorphism between the two domains $k(\mathfrak{p}_i, \alpha)$ and $k(p, \alpha_i^{(r)})$ discussed in the author's paper.

† *Mathematische Annalen*, Vol. 60, Theorem I.

A. If in $E(x)$ $e_i \geqq e_n$, a_n is prime to p and e_n is prime to n, then, in the domain defined by a root of $E(x) = 0$, p is the nth power of a prime divisor and $E(x)$ is irreducible in $k(p)$.

We shall prove the following fact:

B. If in $E(x)$ $e_i \geqq e_{n-1}$, $e_{n-1} \leqq e_n/2$, $n > 2$ and a_{n-1} and a_n are prime to p, then $E(x)$ has a linear factor in $k(p)$.

Since a_{n-1} and a_n are prime to p there exists a c such that $a_{n-1}c + a_n \equiv 0$ mod p. Then

$$E(p^{e_n-e_{n-1}}c) = p^{(e_n-e_{n-1})n}c^n + p^{(e_n-e_{n-1})(n-1)+e_1}a_1c^{n-1} \\ + \cdots + p^{(e_n-e_{n-1})2+e_{n-2}}a_{n-2}c^2 + p^{e_n}(a_{n-1}c + a_n).$$

Since $e_{n-1} \leqq e_n/2$, $e_n - e_{n-1} \geqq e_n/2$; and since $n > 2$, $(e_n - e_{n-1})n > e_n$. Since $e_i \geqq e_{n-1}$ for $i \leqq n-2$, $(e_n - e_{n-1})(n - i) + e_i > e_n - e_{n-1} + e_i > e_n$; and since $a_{n-1}c + a_n \equiv 0$ mod p, we see that

$$E(p^{e_n-e_{n-1}}c) \equiv 0 \text{ mod } p^{e_n+1}.$$

If we next form $E'(p^{e_n-e_{n-1}}c)$, the first term is divisible by $p^{(e_n-e_{n-1})(n-1)}$ and since $e_n - e_{n-1} \geqq n/2$ and $n - 1 \geqq 2$, it is divisible by p^{e_n}. The following terms up to and including the next to the last are divisible by powers of p whose exponents are, in the successive terms, greater than $e_1, e_2, \cdots e_{n-2}$ and hence greater than e_{n-1}. The last term is divisible only by $p^{e_{n-1}}$. Hence $E(x) = 0$ (p) has a solution in $k(p)$,* and in this domain $E(x) \doteq (x - p^{e_n-e_{n-1}}\cdot\gamma)\cdot Q(x)$.

C. If in *B*, $e_{n-1} = 1$, $Q(x)$ is seen to be irreducible by Eisenstein's theorem because it is an Eisensteinian function whose last term contains only the first power of p. If in this case $E(x) = 0$ is irreducible in $k(1)$, p is, in the domain defined by a root of this equation, the product of a prime divisor by the $n - 1$th power of another prime, as is seen by *A* and the isomorphism referred to in the note in the introduction.

We are now ready to consider the factorization of p in the cubic domain $k(\alpha_i)$. I shall denote the prime divisors by $\mathfrak{p}$. As there is no danger of any ambiguity regarding the degrees of the various prime divisors, I shall make no indications of them in the statement of results.

We shall first consider the case when $p > 3$ and is not a divisor of Δ_3.

III. Δ_2 A QUADRATIC RESIDUE MOD p.

Since $\Delta_2 = 81D^2 + 12C^3$ it may happen in this case that $C \equiv 0$ mod p and we shall consider this possibility first.

Since p is not a factor of Δ_3 it is not a factor of the discriminant of $k(\alpha_i)$ and is therefore not divisible by a power of a prime divisor. Moreover the

* Hensel, A.Z., p. 71, bottom.

number of factors of $F_3(x)$ in $k(p)$ is the same as the number of factors of $F_3(x)$ mod p.* When $C \equiv 0$ mod p, $F_3(x) \equiv x^3 + D$ mod p and we see that when $p \equiv 1$ mod 3, if $-D$ is a cubic residue mod p, $F_3(x) \equiv 0$ mod p has three solutions in $k(1)$, and when $-D$ is not a cubic residue the same congruence has no rational solution. If $p \equiv -1$, $-D$ is always a cubic residue mod p but in this case $F_3(x) \equiv 0$ mod p has only one solution in $k(1)$. Hence when $p \equiv +1$ mod 3, if $-D$ is a cubic residue mod p, $F_3(x)$ is in $k(p)$ the product of three linear factors, if $-D$ is not a cubic residue mod p, $F_3(x)$ is irreducible in $k(p)$ and when $p \equiv -1$ mod 3, $F_3(x)$ is the product of a linear and a quadratic factor in $k(p)$.

Since the cubic character of $-D$ is the same as the cubic character of Δ_3 we can conclude that, when $p \equiv +1$ mod 3 if $\Delta_3^{(p-1)/3} \equiv 1$ mod p, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$, and if $\Delta_3^{(p-1)/3} \not\equiv 1$ mod p, then $p \sim \mathfrak{p}$; and when $p \equiv -1$ mod 3, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

If $C \not\equiv 0$ mod p the roots of (1) are units with respect to p. Since Δ_2 is a quadratic residue $F_2(x)$ is reducible in $k(p)$. Hence, if Δ_2 is not a square, $p \sim \mathfrak{p}' \cdot \mathfrak{p}''$ in $k(\mu_1)$ and if Δ_2 is a square, $k(\mu_1) = k(1)$ and $p \sim \mathfrak{p}'$.

Let us now consider the equation

$$(4) \qquad y^3 + \frac{\mu_2}{\mu_1} = 0 \qquad (\mathfrak{p}').$$

If $p \equiv -1$ mod 3 this will always have a solution in $k(\mathfrak{p}', \mu_1)$ but if $p \equiv 1$ mod 3 it has a solution when and only when $(\mu_2/\mu_1)^{(n-1)/3} \equiv 1$ mod $\mathfrak{p}'$. But this condition is equivalent to $\mu_2^{(p-1)/3} - \mu_1^{(p-1)/3} \equiv 0$ mod $\mathfrak{p}'$ and hence according to the notation in the introduction it is seen to be equivalent to $\varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0$ mod $\mathfrak{p}'$. But $\varphi_{(p-1)/3}(-9D, \Delta_2)$ is a rational integer and is therefore divisible by $\mathfrak{p}'$ when and only when it is divisible by p. Hence in order that (4) shall have a solution in $k(\mathfrak{p}', \mu_1)$ it is necessary and sufficient, either that $p \equiv -1$ mod 3, or that $\varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0$ mod p.

Every number of $k(\mathfrak{p}', \mu_1)$ is for the domain of $\mathfrak{p}'$ equal to a number of $k(p)$ and hence there exist rational p-adic numbers m_1 and m_2 such that $\mu_1 = m_1$ $(\mathfrak{p}')$ and $\mu_2 = m_2$ $(\mathfrak{p}')$ and if (4) has a solution this is also for the domain of $\mathfrak{p}'$ equal to a rational p-adic number b.

Let $\overline{\mathfrak{p}}'$ be that prime divisor of $\mathfrak{p}'$ in $k[\mu_1, \sqrt[3]{-(\mu_2/\mu_1)}]$ corresponding to the linear factor of (4). Then $\sqrt[3]{-(\mu_2/\mu_1)} = b$ $(\overline{\mathfrak{p}}')$ and $\overline{\mathfrak{p}}'$ being a factor of $\mathfrak{p}'$, $\mu_1 = m_1$ and $\mu_2 = m_2$ $(\overline{\mathfrak{p}}')$. Hence, if we put $a_0 = (m_1 b + m_2)/(b + 1)$ we see that $\alpha_0 = a_0$ $(\overline{\mathfrak{p}}')$ and hence $F_3(a_0) = 0$ $(\overline{\mathfrak{p}}')$. But from this it follows that $F_3(x) = (x - a_0)Q(x)$ $(\overline{\mathfrak{p}}')$, and since the coefficients of both members of this equation are rational p-adic numbers it is evident that the two members are equal for the domain of $\overline{\mathfrak{p}}'$ when and only when they are equal

* Hensel, A.Z., p. 68.

for the domain of p, and hence

$$F_3(x) = (x - a_0) \cdot Q(x) \quad (p) \tag{5}$$

if (4) has a solution in $k(\mathfrak{p}', \mu_1)$.

If however (4) does not have a solution in $k(\mathfrak{p}', \mu_1)$, it is irreducible in this domain and the same is then also true of $F_3(x)$ and hence a fortiori it is irreducible in $k(p)$.

In regard to $Q(x)$ we observe that since its discriminant differs from Δ_3 only by a square factor it is reducible when Δ_3 is a quadratic residue mod p and is irreducible when Δ_3 is not a quadratic residue mod p. But $\Delta_2 = -3\Delta_3$ is a quadratic residue mod p and hence Δ_3 is a quadratic residue when and only when -3 is a quadratic residue and hence when and only when $p \equiv 1 \bmod 3$.

Hence when $p \equiv 1 \bmod 3$ if $\varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$, then $F_3(x)$ is in $k(p)$ the product of three linear factors and if $\varphi_{(p-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$, $F_3(x)$ is irreducible in $k(p)$; and when $p \equiv -1 \bmod 3$, $F_3(x)$ is in $k(p)$ the product of a linear and a quadratic factor.

Regarding the factors of p in $k(\alpha_i)$ we can conclude that when $p \equiv 1 \bmod 3$ and $\varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$; when $p \equiv 1 \bmod 3$ and $\varphi_{(p-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$, then $p \sim \mathfrak{p}$; and when $p \equiv -1 \bmod 3$, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

IV. Δ_2 NOT A QUADRATIC RESIDUE MOD p.

When Δ_2 is not a quadratic residue mod p, $F_2(x)$ is irreducible in $k(p)$ and hence in $k(\mu_1)$ p is a prime of the second degree. Using the same method as in III we see that the equation

$$y^3 + \mu_2/\mu_1 = 0 \quad (p) \tag{6}$$

has a solution in $k(p, \mu_1)$ when and only when $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$ because in this case it is necessary and sufficient that $(\mu_2/\mu_1)^{(p^2-1)/3} \equiv 1 \bmod p$.

Let us suppose that a solution exists and that $\bar{\mathfrak{p}}$ is the prime divisor of p in $k[\mu_1\sqrt[3]{-(\mu_2/\mu_1)}]$ corresponding to the linear factor of (6). Then there exists an a_0 in $k(p, \mu_1)$ such that $\alpha_0 = a_0(\bar{\mathfrak{p}})$ and hence as above $F_3(x) = (x - a_0)Q(x)$ $(\bar{\mathfrak{p}})$. Again since the coefficients of both members of this equation belong to $k(p, \mu_1)$ in which p is a prime, we see that it is true for the domain of $\bar{\mathfrak{p}}$ when and only when it is true for the domain of p. Hence

$$F_3(x) = (x - a_0)\overline{Q}(x) \quad (p). \tag{7}$$

If a_0 is a rational number, $F_3(x)$ has a linear factor in $k(p)$. If a_0 is a quadratic number and a_0' its conjugate since $F_3(a_0) = 0$ (p), we know that $F_3(a_0') = 0$ (p). Hence $F_3(x) = (x - a_0) \cdot (x - a_0')(x - a)$ (p) and since

the coefficients of $(x - a_0)(x - a_0')$ are rational a must be rational and again we see that $F_3(x)$ has a linear factor in $k(p)$.

If (6) has no solution in $k(p, \mu_1)$ in the same way as in III, we conclude that $F_3(x)$ is irreducible in $k(p)$.

Considering the case when $F_3(x)$ is reducible let us write $F_3(x) = (x - a)\cdot Q(x)$. As in III we conclude that $Q(x)$ is reducible when and only when Δ_3 is a quadratic residue mod p and since now Δ_2 is not a quadratic residue this is the case when and only when -3 is not a quadratic residue and hence $p \equiv -1 \bmod 3$.

Hence when $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$ and $p \equiv -1 \bmod 3$, $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$; when $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$ and $p \equiv 1 \bmod 3$, $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$; and when $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$, $p \sim \mathfrak{p}$.

V. $\Delta_2 \equiv 0$ MOD p, $p > 3$.

We shall next consider the factorization of those primes which are greater than 3 and are factors of the discriminant of $F_3(x)$. We shall write $\Delta_3 = p^s \Delta_3'$ where we suppose that Δ_3' is prime to p. As stated in I we shall put $C = p^\lambda \cdot C_1$, and $D = p^\mu D_1$. Since $s > 0$ if λ or μ is zero the other must be zero also and we shall consider this possibility first. From (2) we have

$$F_3(x) \equiv F_2(x)\cdot\left(x - \frac{3D}{C}\right)\cdot\frac{1}{3C} \bmod p^s \tag{8}$$

and the resultant of the two factors $R = R[F_2(x), x - (3D/C)] = 1/C\cdot(54D^2 - C^3)$. Since $\Delta_3 = 27D^2 + 4C^3 \equiv 0 \bmod p$, $27D^2 \equiv -4C^3 \bmod p$ and hence $R \equiv -9C^2 \bmod p$ and is therefore prime to p. Hence $F_3(x)$ is reducible in $k(p)$.*

We shall again write $F_3(x) \doteq (x - a)Q(x)$ (p) and studying the quadratic factor in the same way as in III we can conclude that when s is even and Δ_3' is a quadratic residue, $Q(x)$ is reducible in $k(p)$, and when s is even and Δ_3' is not a quadratic residue mod p, it is irreducible.

Using the isomorphism mentioned in the note in I we can conclude that when s is even if Δ_3' is a quadratic residue mod p, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$ in $k(\alpha_i)$ and when Δ_3' is not a quadratic residue mod p, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$ in $k(\alpha_i)$. When s is odd, $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$ in $k(\alpha_i)$.

If however $\lambda > 0$ it follows that $\mu > 0$ and we need only consider the cases when $\lambda < 2$ or $\mu < 3$. We shall take up the two cases (a) $\lambda \geqq \mu$ and (b) $\lambda < \mu$.

(a) Since $\lambda \geqq \mu > 0$ by the restrictions imposed on λ and μ we conclude that in this case $\mu = 1$ or 2 and hence by II, A we know that $F_3(x)$ is irreducible in $k(p)$ and $p \sim \mathfrak{p}^3$ in $k(\alpha_i)$.

* Hensel, A.Z., p. 71.

(*b*) Since now $\lambda < \mu$ we see that this is possible only when $\lambda = 1$ and hence, by II, B and C, we can conclude that $F_3(x)$ is in $k(p)$ the product of a linear and a quadratic factor and $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$ in $k(\alpha_i)$.

VI. $p = 3$.

We shall next consider the factorization of 3 in $k(\alpha_i)$. As before we shall write $C = 3^\lambda C_1$, $D = 3^\mu D_1$ and $\Delta_3 = 3^s \Delta_3'$. If $\lambda = 0$, $s = 0$ and we need only consider $F_3(x)$ with respect to the modulus 3. There are six possible forms as follows:

$$\begin{array}{ll}
F_3(x) \equiv x^3 + x \equiv x(x^2+1); & \Delta_3 \equiv -1 \bmod 3, \\
F_3(x) \equiv x^3 + 2x \equiv x(x-1)(x+1); & \Delta_3 \equiv 1 \bmod 3, \\
F_3(x) \equiv x^3 + x + 1 \equiv (x-1)(x^2+x+2); & \Delta_3 \equiv -1 \bmod 3, \\
F_3(x) \equiv x^3 + 2x + 1 \text{ irreducible}; & \Delta_3 \equiv 1 \bmod 3, \\
F_3(x) \equiv x^3 + x + 2 \equiv (x-2)(x^2+2x+2); & \Delta_3 \equiv -1 \bmod 3, \\
F_3(x) \equiv x^3 + 2x + 2 \text{ irreducible}; & \Delta_3 \equiv 1 \bmod 3.
\end{array}$$

From a consideration of this table we note that when $\Delta_3 \equiv 1 \bmod 3$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$ if $D \equiv 0 \bmod 3$ and $3 \sim \mathfrak{p}$ if $D \not\equiv 0 \bmod 3$; and when $\Delta_3 \equiv -1 \bmod 3$, then $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

If $s > 0$ it is necessary that $\lambda > 0$. As before we need only consider the cases when $\lambda < 2$ or $\mu < 3$. These we shall consider as follows (*a*) $\lambda = 1$, $\mu = 0$; (*b*) $\lambda > 1$, $\mu = 0$; (*c*) $\lambda \geqq \mu$; (*d*) $\lambda < \mu$.

(*a*) When $\lambda = 1$, $\mu = 0$, $\Delta_3 = -27(D^2 + 4C_1^3)$ and $s \geqq 3$. The function $F_2(x)$ is $9C_1x^2 + 9Dx - 9C_1^2$ and μ_1 and μ_2 are roots of

$$C_1x^2 + Dx - C_1^2 = 0. \tag{9}$$

In place of the equation (2) we have now

$$F_3(x) = (C_1x^2 + Dx - C_1^2)\left(x - \frac{D}{C_1}\right) \cdot \frac{1}{C_1} - \frac{\Delta_3}{27C_1}x \tag{10}$$

and hence when $s > 3$,

$$F_3(x) \equiv (C_1x^2 + Dx - C_1^2)\left(x - \frac{D}{C_1}\right) \cdot \frac{1}{C_1} \bmod 3^{s-3}. \tag{11}$$

Since in this case (i.e., $s > 3$) $D^2 + 4C_1^3 \equiv 0 \bmod 3^{s-3}$,

$$D^2 \equiv -4C_1^3 \bmod 3^{s-3}. \tag{12}$$

Therefore

$$F_3'\left(\frac{D}{C_1}\right) \equiv \frac{3D^2 + 3C_1^3}{C_1^2} = \frac{3}{C_1^2}(D^2 + C_1^3) \equiv -9C_1 \bmod 3^{s-3}$$

and if $s > 6$, $F_3'(D/C_1)$ is divisible only by 3^2. $F_3''(D/C_1)/2! = 3(D/C_1)$ and is divisible only by 3. $F_3'''(D/C_1)/3! = 1$ and is prime to 3. Hence using the theorem on pages 73 and 74 of Hensel's Theorie der Algebraischen

Zahlen we see that when $s > 6$, $F_3(x)$ has a linear factor in $k(3)$, and shall write

$$F_3(x) = (x - a)Q(x)\cdot(3).$$

Regarding the quadratic factor we can conclude in the same way as in V that it is reducible when and only when s is even and $\Delta_3' \equiv 1 \bmod 3$. By the isomorphism previously used we then see that if s is even, when $\Delta_3' \equiv 1 \bmod 3$, $3 \sim \mathfrak{p}_1\cdot\mathfrak{p}_2\cdot\mathfrak{p}_3$, and when $\Delta_3' \equiv -1 \bmod 3$, $3 \sim \mathfrak{p}_1\cdot\mathfrak{p}_2$; and that when s is odd, $3 \sim \mathfrak{p}_1\cdot\mathfrak{p}_2^2$.

If $s = 6$, let us consider first the case when Δ_3' is not a quadratic residue mod 3. Then

$$\Delta_3' = -\frac{D^2 + 4C_1^3}{27} \equiv -1 \bmod 3 \tag{13}$$

and since $D^2 + 4C_1^3 \equiv 0 \bmod 3$ and $D^2 \equiv 4 \equiv 1 \bmod 3$, $C^3 \equiv -1 \bmod 3$ and we can write (13) in the form

$$-\frac{D^2 + 4C_1^3}{27} \equiv C_1^3 \bmod 3$$

whence $-D^2 - 4C_1^3 \equiv 27C_1^3 \bmod 81$ or

$$D^2 \equiv -31C_1^3 \bmod 81. \tag{14}$$

Now

$$F_3\left(-\frac{2D}{C_1}\right) = -\frac{D}{C_1^3}(8D^2 + 5C_1^3) \equiv 243D \equiv 0 \bmod 81$$

by (14).

$$F_3'\left(-\frac{2D}{C_1}\right) = \frac{3}{C_1^2}(4D^2 + C_1^3) \equiv -123\cdot 3C_1 \bmod 81,$$

and is therefore divisible only by 3^2.

$$F_3''\left(-\frac{2D}{C_1}\right)\Big/2! = -\frac{6D}{C_1}$$

and is divisible only by 3.

$$F_3'''\left(-\frac{2D}{C_1}\right)\Big/3! = 1$$

and is prime to 3. In the same way as above we can conclude that $F_3(x)$ has a linear factor in $k(3)$ and since Δ_3' is not a quadratic residue the quadratic factor is irreducible in $k(3)$ and as above we conclude that $3 \sim \mathfrak{p}_1\cdot\mathfrak{p}_2$ in $k(\alpha_i)$.

If however $s = 6$ and Δ_3' is a quadratic residue mod 3, (10) shows that $F_3(D/C_1)$ is divisible by 3^3 but not by 3^4. Since now

$$\Delta_3' = -\frac{D^2 + 4C_1^3}{27} \equiv 1 \equiv -C_1^3 \bmod 3,$$

$-D^2 - 4C_1^3 \equiv -27C_1^3 \bmod 81$ and

$$D^2 \equiv 23C_1^3 \bmod 81. \tag{15}$$

Let us now form

$$F_3\left(3y + \frac{D}{C_1}\right) = 27y^3 + 27\frac{D}{C_1}y^2 + 9\left(\frac{D^2}{C_1^2} + C_1\right)y + F_3\left(\frac{D}{C_1}\right).$$

Dividing by 27 and equating to zero we have

$$y^3 + \frac{D}{C_1}y^2 + \frac{D^2 + C_1^3}{3C_1^2}y + F_3\left(\frac{D}{C_1}\right)\Big/27 = 0 \tag{16}$$

and evidently one root of this equation belongs to $k(\alpha_i)$. In this equation D/C_1 and $F_3(D/C_1)/27$ are known to be prime to 3. By (15) $D^2 + C_1^3 \equiv 24C_1^3$ mod 81 and hence $(D^2 + C_1^3)3C_1^2 \equiv 8C_1 \bmod 27$ and is therefore also prime to 3.

We shall next see that the left-hand member of (16) is irreducible mod 3. Suppose that it is not. It must then have a linear factor and hence there must exist an a such that

$$a^3 + \frac{D}{C_1}a^2 + \frac{D^2 + C_1^3}{3C_1^2}a + F_3\left(\frac{D}{C_1}\right)\Big/27 \equiv 0 \bmod 3.$$

Since the last term is prime to 3, a cannot be divisible by 3 and hence $a^3 \equiv a \bmod 3$ and $a^2 \equiv 1 \bmod 3$ and a must satisfy

$$\left(1 + \frac{D^2 + C_1^3}{3C_1^2}\right)a + \frac{D}{C_1} + F_3\left(\frac{D}{C_1}\right)\Big/27 \equiv 0 \bmod 3. \tag{17}$$

We have seen that $(D^2 + C_1^3)/3C_1^2 \equiv 8C_1 \bmod 27$ and since $C_1^3 \equiv -1 \bmod 3$, $C_1 \equiv -1 \bmod 3$ and hence $(D^2 + C_1^3)/3C_1^2 \equiv -8 \equiv 1 \bmod 3$.

$$\frac{D}{C_1} + F_3\left(\frac{D}{C_1}\right)\Big/27 = \frac{D}{27C_1^3}[27C_1^2 + D^2 + 4C_1^3].$$

By 15 we have $27C_1^2 + D^2 + 4C_1^3 \equiv 27C_1^2 + 27C_1^3 \bmod 81$ and hence

$$\frac{D}{27C_1^3}[27C_1^2 + D^2 + 4C_1^3] \equiv \frac{D}{C_1}(1 + C_1) \equiv 0 \bmod 3.$$

The congruence (17) therefore reduces to $2a \equiv 0 \bmod 3$ which is impossible since a is prime to 3. The left-hand member of (16) is therefore irreducible mod 3 and hence also in $k(3)$. Since this is so we know that 3 cannot be a divisor of the discriminant of (16) and hence also not a divisor of the discriminant of $k(\alpha_i)$ and is therefore not divisible by a power of a prime divisor and hence $3 \sim \mathfrak{p}$ in $k(\alpha_i)$.

If $s = 5$, $F_3(D/C_1)$ is divisible by 9 but not by 27. Multiplying (16) by 3 and forming the equation whose roots are the reciprocals of the roots of (16) we have

$$\frac{F_3(D/C_1)}{9} z^3 + \frac{D^2 + C_1^3}{C_1^2} z^2 + \frac{3D}{C_1} z + 3 = 0$$

and by II, A we see that this is irreducible in $k(p)$ and that in $k(\alpha_i)$ $3 \sim \mathfrak{p}^3$.

If $s = 4$, $F_3(D/C_1)$ is divisible only by 3 and hence the equation

$$F\left[y + \left(\frac{D}{C_1}\right)\right] = 0$$

is seen by II, A to be irreducible in $k(3)$ and again $3 \sim \mathfrak{p}^3$ in $k(\alpha_i)$.

If $s = 3$, $\Delta'_3 = -D^2 - 4C_1^3$ and the discriminant of (9) is $-\Delta'_3$ Since $D^2 \equiv 1 \bmod 3$ and $\Delta'_3 \not\equiv 0 \bmod 3$ it is necessary that $C \equiv 1 \bmod 3$ and hence $-\Delta'_3 \equiv 2 \bmod 3$. The equation (9) is therefore irreducible in $k(3)$ and in $k(\mu_1)$ 3 is a prime of the second degree. As in IV we can show that $F_3(x)$ is reducible in $k(3)$ when and only when the equation $y^3 + (\mu_2/\mu_1) = 0$ (3) has a solution in $k(3, \mu_1)$. This equation has a solution when and only when the congruence $y^3 + (\mu_2/\mu_1) \equiv 0 \bmod 9$ has a solution in the given domain.*

If the given congruence has a solution b, we see that $b^9 \equiv -(\mu_2/\mu_1)^3$ mod 27 and since every integer of $k(3, \mu_1)$ satisfies the congruence $x^9 \equiv x$ mod 3, we conclude that $b \equiv -(\mu_2/\mu_1)^3 \bmod 3$. Hence if $y^3 + (\mu_2/\mu_1) \equiv 0$ mod 9 has a solution, there must exist a c such that $[3c - (\mu_2/\mu_1)^3]^3 + \mu_2/\mu_1 \equiv 0 \bmod 9$. By expanding we find that the necessary and sufficient condition for the reducibility of $F_3(x)$ is $-(\mu_2/\mu_1)^9 + \mu_2/\mu_1 \equiv 0 \bmod 9$, or $(\mu_2/\mu_1)^8 \equiv 1 \bmod 9$ which reduces to $\varphi_8(-D, -\Delta'_3) \equiv 0 \bmod 9$.

But $\varphi_8(-D, -\Delta'_3) = -8D^7 + 56D^5\Delta'_3 - 56D^3\Delta'^2_3 + 8D\Delta'^3_3$ and hence since $8D$ is relatively prime to 3, the necessary and sufficient condition is $\Delta'^3_3 - 7\Delta'^2_3 D^2 + 7\Delta'_3 D^4 - D^6 \equiv 0 \bmod 9$ or in another form

$$\left(\frac{\Delta'_3}{D^2}\right)^3 - 7\left(\frac{\Delta'_3}{D^2}\right)^2 + 7\left(\frac{\Delta'_3}{D^2}\right) - 1 \equiv 0 \bmod 9. \tag{18}$$

But 1 is the only rational solution of $x^3 - 7x^2 + 7x - 1 \equiv 0 \bmod 9$ and hence (18) is possible when and only when $\Delta'_3/D^2 \equiv 1 \bmod 9$, or $-D^2 - 4C_1^3 \equiv D^2 \bmod 9$. This reduces to $D^2 + 2C_1^3 \equiv 0 \bmod 9$ and since $C_1 \equiv 1 \bmod 3$, $C_1^3 \equiv 1 \bmod 9$ and hence this condition is equivalent to $D^2 + 2 \equiv 0 \bmod 9$.

Hence $F_3(x)$ is reducible in $k(3)$ when and only when $D^2 + 2 \equiv 0 \bmod 9$. Since $s = 3$, 3 is in $k(\alpha_i)$ divisible by a power of a prime divisor and hence $F_3(x)$ cannot have three linear factors in $k(3)$. We can therefore conclude that when $D^2 + 2 \equiv 0 \bmod 9$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$, and when $D^2 + 2 \not\equiv 0 \bmod 9$, $s \sim \mathfrak{p}^3$.

* Author, *Crelle's Journal*, Vol. 145.

(*b*) Since in this case $\lambda > 1$, $\mu = 0$, we have $s = 3$ and again $F_3(x)$ is either irreducible or the product of a linear and a quadratic factor in $k(3)$. If $D^2 \equiv 1 \bmod 9$, $F_3(-D) \equiv 0 \bmod 9$ and hence $F_3(x-D)$ is by II, B seen to be reducible and hence $F_3(x)$ is in $k(3)$ the product of a linear and a quadratic factor. If $D^2 \not\equiv 1 \bmod 9$, $F_3(-D)$ is divisible by 3 but not by 9 and hence by II, A, $F_3(x-D)$ is seen to be irreducible in $k(3)$.

Hence when $D^2 \equiv 1 \bmod 9$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$ and when $D^2 \not\equiv 1 \bmod 9$, $3 \sim \mathfrak{p}^3$.

(*c*) In this case $\lambda \geqq \mu > 0$ and hence by the restrictions imposed on λ and μ we conclude that $\mu = 1$ or 2 and by II, A, $F_3(x)$ is irreducible in $k(3)$ and in $k(\alpha_i)$, $3 \sim \mathfrak{p}^3$.

(*d*) In this case $\lambda < \mu$ and again by the restrictions on λ and μ we see that this is possible only when $\lambda = 1$ and by II, B we conclude that $F_3(x)$ is in $k(3)$ the product of a linear and a quadratic factor and in $k(\alpha_i)$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$.

VII. $p = 2$.

We observe that when $p = 2$ if D is odd $s = 0$ and Δ_3 is not divisible by 2 and we need only consider $F_3(x)$ relative to the modulus 2. Two possibilities occur as follows:

$$F_3(x) \equiv x^3 + x + 1 \text{ irreducible mod } 2,$$
$$F_3(x) \equiv x^3 + 1 \equiv (x-1)(x^2 + x + 1) \bmod 2.$$

Hence when C is odd, $F_3(x)$ is irreducible in $k(2)$ and in $k(\alpha_i)$, $2 \sim \mathfrak{p}$; and when C is even, $F_3(x)$ is in $k(2)$ the product of a linear and a quadratic factor in $k(2)$ and in $k(\alpha_i)$, $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

If $s > 0$ it is necessary that $\mu > 0$ and we shall consider the cases (*a*) $\lambda = 0$, $\mu \geqq 1$; (*b*) $\lambda \geqq \mu$; (*c*) $\lambda < \mu$.

(*a*) In this case $s \geqq 2$. We have $F_3(x) \equiv x(x^2 + C) \bmod 2^\mu$ and since $R(x, x^2 + C) = C$ is odd we conclude that in this case $F_3(x)$ has a factor in $k(2)$* and we shall write $F_3(x) = (x - a)Q(x)$. Using the same reasoning as before we see that when s is even and $\Delta_3' \equiv 1 \bmod 8$, then the quadratic factor is reducible and, in all other cases, irreducible.

We can then conclude by making use of the isomorphism previously referred to and the factorization of 2 in a quadratic domain† that when s is even, if $\Delta_3' \equiv 1 \bmod 8$, then $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$, and if $\Delta_3' \not\equiv 1 \bmod 8$ but $\Delta_3' \equiv 1 \bmod 4$, then $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$, and if $\Delta_3' \equiv 3 \bmod 4$ or when s is odd, $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$.

(*b*) Since in this case $\lambda \geqq \mu$ as in IV, (*c*), we can conclude that $\mu = 1$ or 2 and by II, A, $F_3(x)$ is irreducible in $k(2)$ and $2 \sim \mathfrak{p}^3$ in $k(\alpha_i)$.

(*c*) Since in this case $\lambda < \mu$ we know again that $\lambda = 1$ and again by II, B and C, $F_3(x)$ is in $k(2)$ the product of a linear and a quadratic factor and in $k(\alpha_i)$, $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$.

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* Hensel, A.Z., p. 71.

† Hilbert, Report, *Jahresbericht der Deut. Math. ver.*, Vol. 4.

SUMMARY.

- $p > 3$
  - $\Delta_2^{(p-1)/2} \equiv 1 \bmod p$
    - $p \equiv 1 \bmod 3$

      $$\left.\begin{array}{ll} C \equiv 0 \bmod p, & \Delta_3^{(p-1)/3} \equiv 1 \bmod p \\ C \not\equiv 0 \bmod p, & \varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p \end{array}\right\} p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$$

      $$\left.\begin{array}{ll} C \equiv 0 \bmod p, & \Delta_3^{(p-1)/3} \not\equiv 1 \bmod p \\ C \not\equiv 0 \bmod p, & \varphi_{(p-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p \end{array}\right\} p \sim \mathfrak{p}$$
    - $p \equiv -1 \bmod 3$ . . . . $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$
  - $\Delta_2^{(p-1)/2} \equiv -1 \bmod p$
    - $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$
      - $p \equiv -1 \bmod 3$ . . . . $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$
      - $p \equiv 1 \bmod 3$ . . . . $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$
    - $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$ . . . . $p \sim \mathfrak{p}$
  - $\Delta_2 \equiv 0 \bmod p, \quad \Delta_3 = p^s \Delta_3'$
    - $\lambda = \mu = 0$
      - s even
        - $(\Delta_3')^{(p-1)/2} \equiv 1 \bmod p$ . . . . $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$
        - $(\Delta_3')^{(p-1)/2} \equiv -1 \bmod p$ . . . . $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$
      - s odd . . . . $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$
    - $\lambda \geqq \mu > 0$ . . . . $p \sim \mathfrak{p}^3$
    - $0 < \lambda < \mu$ . . . . $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$
- $p = 3$
  - $\lambda = 0$
    - $\Delta_3 \equiv 1 \bmod 3$
      - $D \equiv 0 \bmod 3$ . . . . $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$
      - $D \not\equiv 0 \bmod 3$ . . . . $3 \sim \mathfrak{p}$
    - $\Delta_3 \equiv 2 \bmod 3$ . . . . $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$
  - $\lambda = 1, \quad \mu = 0$
    - $s \geqq 7$
      - s even
        - $\Delta_3' \equiv 1 \bmod 3$ . . . . $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$
        - $\Delta_3' \equiv -1 \bmod 3$ . . . . $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$
      - s odd . . . . $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$
    - $s = 6$
      - $\Delta_3' \equiv 1 \bmod 3$ . . . . $3 \sim \mathfrak{p}$
      - $\Delta_3' \equiv -1 \bmod 3$ . . . . $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$
    - $s = 5$ or 4 . . . . $3 \sim \mathfrak{p}^3$
    - $s = 3$
      - $D^2 + 2 \equiv 0 \bmod 9$ . . . . $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$
      - $D^2 + 2 \not\equiv 0 \bmod 9$ . . . . $3 \sim \mathfrak{p}^3$

$\lambda > 1, \quad \mu = 0$

$D^2 \equiv 1 \bmod 9$ .... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$D^2 \not\equiv 1 \bmod 9$ .... $3 \sim \mathfrak{p}^3$

$\lambda \geqq \mu > 0$ .... $3 \sim \mathfrak{p}^3$

$\lambda < \mu$ .... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$p = 2$

$\mu = 0$

$C \equiv 0 \bmod 2$ .... $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

$C \not\equiv 0 \bmod 2$ .... $2 \sim \mathfrak{p}$

$\lambda = 0, \quad \mu \geqq 1$

s even

$\Delta_3' \equiv 1 \bmod 8$ .... $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$

$\Delta_3' \not\equiv 1 \bmod 8$

$\Delta_3' \equiv 1 \bmod 4$ .... $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

$\Delta_3' \equiv 3 \bmod 4$ .... $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

s odd .... $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$\lambda \geqq \mu > 0$ .... $2 \sim \mathfrak{p}^3$

$\lambda < \mu$ .... $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

ON THE STRUCTURE OF FINITE CONTINUOUS GROUPS WITH A SINGLE EXCEPTIONAL INFINITESIMAL TRANSFORMATION.

By S. D. Zeldin.

Introduction.—Let $X_1, \cdots, X_r, X_{r+1}$ be the symbols of the infinitesimal transformations of a finite continuous group G with $r+1$ essential parameters, in which case

$$(1) \qquad (X_i, X_j) = \sum_{k=1}^{r+1} c_{ijk} X_k{}^* \quad (i, j = 1, 2, \cdots, r+1).$$

The symbols of the infinitesimal transformations of the group Γ adjoint to G are then

$$(2) \qquad D_i = \sum_{j=1}^{r+1} \sum_{k=1}^{r+1} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \dagger \qquad (i = 1, 2, \cdots, r+1);$$

and we have

$$(3) \qquad (D_i, D_j) = \sum_{k=1}^{r+1} c_{ijk} D_k \ddagger \quad (i, j = 1, 2, \cdots, r+1).$$

The group Γ has $r+1$ essential parameters (i.e., there is no linear relation with constant coefficients between $D_1, \cdots, D_{r+1}$) if, and only if, G contains no exceptional§ infinitesimal transformation, i.e., no infinitesimal transformation $\sum_{i=1}^{r+1} a_i X_i$ such that

$$(4) \qquad (\sum_{i=1}^{r+1} a_i X_i, X_j) = \sum_{i=1}^{r+1} a_i (X_i, X_j) = 0 \quad (j = 1, 2, \cdots, r+1).$$

Corresponding to each independent exceptional infinitesimal transformation of G, $\sum a_i X_i$, is an independent linear relation $\sum_{i=1}^{r+1} a_i D_i = 0$ between the differential operators $D_1, \cdots, D_r, D_{r+1}$.||

We shall assume in what follows that the group G has just one exceptional infinitesimal transformation, which we may take without loss of generality to be X_{r+1}. In this case,

$$(5) \qquad \begin{aligned} (X_i, X_j) &= \sum_{k=1}^{r} c_{ijk} X_k + c_{i,\, j,\, r+1} X_{r+1} \quad && (i, j = 1, 2, \cdots, r), \\ (X_i, X_{r+1}) &= (X_{r+1}, X_i) = 0 && (i = 1, 2, \cdots, r, r+1); \end{aligned}$$

* Lie-Scheffers, Continuirliche Gruppen, p. 391.

† Lie, loc. cit., p. 466.

‡ Lie, loc. cit., p. 467.

§ Lie calls it "ausgezeichnete," loc. cit., p. 465.

|| Lie, loc. cit., p. 465.

so that

$$(6)\qquad c_{i,\,r+1,\,k} = -c_{r+1,\,i,\,k} = 0 \quad (i,k = 1, 2, \cdots, r, r+1);$$

and, therefore,

$$(7)\qquad D_{r+1} \doteq 0, \qquad D_i = \sum_{j=1}^{r}\sum_{k=1}^{r+1} \alpha_j c_{jik}\frac{\partial}{\partial \alpha_k}, \qquad (D_i, D_j) = \sum_{k=1}^{r} c_{ijk} D_k \quad (i, j = 1, \cdots, r).$$

In consequence of the assumption that G contains just one exceptional transformation, there will exist a finite continuous group G' with r essential parameters generated by r infinitesimal transformations, whose symbols we shall denote by $Y_1, \cdots, Y_r$, such that

$$(8)\qquad (Y_i, Y_j) = \sum_{k=1}^{r} c_{ijk} Y_k^{*} \qquad (i, j = 1, 2, \cdots, r).$$

Let Γ' denote the adjoint of G', and the symbols of its infinitesimal transformations let be $D'_1, \cdots, D'_r$, where

$$(9)\qquad D'_i \equiv \sum\sum \alpha_j c_{jik}\frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r).$$

We have then

$$(10)\qquad (D'_i, D'_j) = \sum_{k=1}^{r} c_{ijk} D'_k \qquad (i, j = 1, 2, \cdots, r).$$

In what follows we shall investigate the conditions that may be imposed on the structure of G' (when, as assumed, G has just one exceptional infinitesimal transformation) so that we shall have

$$(11)\qquad c_{i,\,k,\,r+1} = -c_{k,\,i,\,r+1} = 0 \quad (i, k = 1, \cdots, r, r+1);$$

and we shall show, if there is just one spread invariant to the adjoint of G' (which will be assumed in what follows), that then

$$(11)\qquad c_{i,\,k,\,r+1} = -c_{k,\,i,\,r+1} = 0 \quad (i, k = 1, \cdots, r, r+1),$$

or, by a suitable choice of the X's, these conditions will be satisfied. From what is stated below, it follows that, when the adjoint of G' has but one invariant spread, this spread is an $(r-1)$-spread.

Invariant spreads of the adjoint.—Lie shows† that the invariants of the adjoint of any finite continuous group may be taken homogeneous: if there is only one invariant, it will be homogeneous but not of order zero; if there are two or more invariants, all of them can be taken homogeneous and of order zero except one, which will be homogeneous but not of order zero.

* Lie-Engel, "Transformations Gruppen," Vol. I, pp. 300–305.

† Lie-Scheffers, "Continuirliche Gruppen," pp. 596–599.

Every function of $\alpha_1, \cdots, \alpha_r$ invariant to the adjoint Γ' of G', therefore, yields a spread invariant to Γ'. Since, by supposition, the adjoint of G' has but one invariant spread, there is therefore but one function of $\alpha_1, \cdots, \alpha_r$ invariant to the adjoint of G'. This function will be denoted, in this paper, by $\varphi \equiv \varphi(\alpha_1, \cdots, \alpha_r)$, homogeneous in the α's; and then $\varphi(\alpha) = 0$ will be the only spread invariant to the adjoint of G'.

The function $\varphi(\alpha)$ is a solution of the system of equations

$$\begin{aligned}(12)\qquad D'_i f(\alpha) \equiv \sum_{j=1}^{r} \alpha_j c_{ji1} \frac{\partial f(\alpha)}{\partial \alpha_1} \\ + \cdots + \sum_{j=1}^{r} \alpha_j c_{jir} \frac{\partial f(\alpha)}{\partial \alpha_r} = 0 \quad (i = 1, 2, \cdots, r).\end{aligned}$$

From $\partial\varphi(\alpha)/\partial\alpha_{r+1} = 0$ it follows, by (7), that

$$\begin{aligned}(13)\qquad D_i\varphi(\alpha) \equiv \sum_{j=1}^{r} \alpha_j c_{ji1} \frac{\partial \varphi(\alpha)}{\partial \alpha_1} \\ + \cdots + \sum_{j=1}^{r} \alpha_j c_{jir} \frac{\partial \varphi(\alpha)}{\partial \alpha_r} \equiv D'_i\varphi(\alpha) = 0 \quad (i = 1, 2, \cdots, r);\end{aligned}$$

therefore, $\varphi(\alpha)$ is also an invariant of the adjoint of G.

It is convenient to denote by

$$(14)\qquad E_i \equiv \begin{vmatrix} c_{i11}, & \cdots, & c_{ir1}, & 0 \\ c_{i12}, & \cdots, & c_{ir2}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{i1r}, & \cdots, & c_{irr}, & 0 \\ c_{i,1,r+1}, & \cdots, & c_{i,r,r+1}, & 0 \end{vmatrix} \qquad (i = 1, 2, \cdots, r)$$

the matrix of the differential operators

$$D_i \equiv \sum_{j=1}^{r} \sum_{k=1}^{r+1} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r).$$

Then

$$(15)\qquad \sum a_i E_i \equiv \begin{vmatrix} \sum a_i c_{i,1,1}, & \cdots, & \sum a_i c_{i,r,1}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sum a_i c_{i,1,r}, & \cdots, & \sum a_i c_{i,r,r}, & 0 \\ \sum a_i c_{i,1,r+1}, & \cdots, & \sum a_i c_{i,r,r+1}, & 0 \end{vmatrix}$$

is the matrix of the general infinitesimal transformation $\sum_{i=1}^{r} a_i D_i$ of Γ. Similarly we shall write

$$(16)\qquad \mathcal{E}_i \equiv \begin{vmatrix} c_{i11}, & \cdots, & c_{ir1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{i1r}, & \cdots, & c_{irr} \end{vmatrix} \qquad (i = 1, 2, \cdots, r)$$

to denote the matrix of the differential operators

$$D'_i \equiv \sum_{j=1}^{r}\sum_{k=1}^{r} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r);$$

and then

$$\sum_1^r a_i \mathcal{E}_i \equiv \begin{vmatrix} \sum a_i c_{i11}, & \cdots, & \sum a_i c_{ir1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \sum a_i c_{i1r}, & \cdots, & \sum a_i c_{irr} \end{vmatrix} \tag{17}$$

will denote the matrix of the general infinitesimal transformation $\sum_{i=1}^r a_i D'_i$ of Γ'.

Any invariant of the adjoint of G being a solution of the complete system of equations

$$D_i f(\alpha) \equiv \sum_{j=1}^{r+1}\sum_{k=1} \alpha_j c_{jik} \frac{\partial f(\alpha)}{\partial \alpha_k} = 0 \qquad (i = 1, 2, \cdots, r), \tag{18}$$

the number of invariants of the group Γ, adjoint to G, is determined by the array

$$\begin{matrix} \sum_1^r \alpha_j c_{j11}, & \cdots, & \sum_1^r \alpha_j c_{j1r}, & \sum \alpha_j c_{j,\,1,\,r+1} \\ \cdot & \cdot & \cdot & \cdot \\ \sum \alpha_j c_{jr1}, & \cdots, & \sum \alpha_j c_{jrr}, & \sum \alpha_j c_{j,\,r,\,r+1} \end{matrix} \tag{19}$$

being equal to $r + 1$ less the order of the non-zero determinant of highest order formed from the array, and thus to the nullity of $\sum \alpha_j \breve{E}_j$, where $\breve{E}_j$ denotes the transverse of E_j. Therefore, the number of invariants of G is equal to the nullity of $\sum \alpha_j E_j$, being a solution of the complete system of equations

$$D'_i f(\alpha) \equiv \sum_{j=1}^{r}\sum_{k=1}^{r} \alpha_j c_{jik} \frac{\partial f(\alpha)}{\partial \alpha_k} = 0 \qquad (i = 1, 2, \cdots, r). \tag{20}$$

The number of invariants of the adjoint of G' is equal to the nullity of the array

$$\begin{matrix} \sum_1^r \alpha_j c_{j11}, & \cdots, & \sum_1^r \alpha_j c_{j1r} \\ \cdot & \cdot & \cdot \\ \sum_1^r \alpha_j c_{jr1}, & \cdots, & \sum_1^r \alpha_j c_{jrr} \end{matrix} \tag{21}$$

and thus is equal to the nullity of $\sum_1^r \alpha_j \breve{\mathcal{E}}_j$ (where, as before, $\breve{\mathcal{E}}_j$ is the transverse of $\mathcal{E}_j$), or, what is the same thing, to the nullity of $\sum_1^r \alpha_j \mathcal{E}_j$.

Since it is assumed that the adjoint of G' has just one invariant, it follows that the nullity of $\sum_1^r \alpha_i \mathcal{E}_i$ is one for an arbitrary system of values

of the α's, i.e., at least one of the minors of $|\sum \alpha_i \mathcal{E}_i|$* of order $r-1$ is not zero. But every minor of $|\sum \alpha_i \mathcal{E}_i|$ is a minor of $|\sum \alpha_i E_i|$, therefore at least one minor of the order $r-1$ of the determinant $|\sum \alpha_i E_i|$ is not zero, and thus the nullity of the matrix $\sum \alpha_i E_i$ *cannot exceed two* for an arbitrary system of values of the α's.

Further, for $\alpha_1, \cdots, \alpha_r, \alpha_{r+1}$ assigned, the symbolic equation

$$\left(\sum_1^{r+1} \alpha_i X_i,\ \sum_1^{r+1} \eta_i X_i\right) = 0, \tag{22}$$

is satisfied for

$\eta_1 = \alpha_1, \cdots, \eta_r = \alpha_r, \eta_{r+1} = 0$ and $\eta_1 = 0, \cdots, \eta_r = 0, \eta_{r+1} = 1.$

But from (22) follows

$$\sum \alpha_i E_i(\eta_1, \cdots, \eta_{r+1}) = \begin{pmatrix} \sum \alpha_i c_{i11}, & \cdots \\ \vdots & \\ \sum \alpha_i c_{i,1,r+1}, & \cdots \end{pmatrix} (\eta_1, \cdots, \eta_{r+1}) = 0.\dagger \tag{23}$$

Therefore, this system of linear equations has two independent solutions, namely,

$\eta_1 = \alpha_1, \cdots, \eta_r = \alpha_r, \eta_{r+1} = 0$ and $\eta_1 = 0, \cdots, \eta_r = 0, \eta_{r+1} = 1.$

Whence it follows that the nullity of the matrix $\sum \alpha_i E_i$, for any system of values of the α's, is at least two. Wherefore, the nullity of $\sum \alpha_i E_i$, for an arbitrary system of values of the α's, is just *equal to two;* and thus the number of *invariants* of the adjoint of G is *equal to two.*

The Holomorphic Invariant of the Adjoint.—Let the second invariant of the adjoint of G, independent of $\varphi(\alpha)$, be $\psi \equiv \psi(\alpha, \cdots, \alpha_{r+1})$ homogeneous in the α's. We must have

$$\frac{\partial \psi(\alpha)}{\partial \alpha_{r+1}} \not\equiv 0. \tag{24}$$

For, otherwise, if $\dfrac{\partial \psi(\alpha)}{\partial \alpha_{r+1}} \equiv 0$, we get

$$D'_i \psi(\alpha) \equiv D_i \psi(\alpha) \equiv 0, \tag{25}$$

and then, since the group adjoint to G' has but one invariant, we should have $\psi = W(\varphi)$, which is contrary to our assumption.

By Lie's theorem, as stated above, $\psi(\alpha)$ can be taken homogeneous (and of order zero, since a homogeneous invariant of the adjoint of G, viz., $\varphi(\alpha_1, \cdots, \alpha_r)$, not of order zero, exists); and thus, we may put

$$\psi(\alpha) \equiv F\left(\frac{\alpha_1}{\alpha_{r+1}}, \cdots, \frac{\alpha_r}{\alpha_{r+1}}\right). \tag{26}$$

* The determinant of any matrix M will be denoted by $|M|$.

† Lie-Scheffers, "Continuirliche Gruppen," pp. 558, 562. The notation used here is due to Cayley, *Philosophical Transactions*, 1858.

Let

$$\beta_1 = \frac{\alpha_1}{\alpha_{r+1}}, \cdots, \beta_r = \frac{\alpha_r}{\alpha_{r+1}}, \beta_{r+1} = \alpha_{r+1}; \tag{27}$$

then

$$\alpha_1 = \beta_1 \beta_{r+1}, \cdots, \alpha_r = \beta_r\beta_{r+1}, \alpha_{r+1} = \beta_{r+1} \tag{28}$$

and

$$\frac{\partial F(\beta)}{\partial \beta_{r+1}} \equiv 0. \tag{29}$$

Let

$$\begin{aligned} \overline{D}_i \equiv{}& D_i\beta_1 \frac{\partial}{\partial \beta_1} + \cdots + D_i\beta_r \frac{\partial}{\partial \beta_r} + D_i\beta_{r+1} \frac{\partial}{\partial \beta_{r+1}} \\ \equiv{}& \left(\sum_{j=1}^{r} c_{ji1}\beta_j - \beta_1 \sum_{j=1}^{r} c_{jir+1}\right) \frac{\partial}{\partial \beta_1} \\ &+ \cdots + \left(\sum_{j=1}^{r} c_{jir}\beta_j - \beta_r \sum_{1}^{r} c_{jir+1}\beta_j\right) \frac{\partial}{\partial \beta_r} \\ &+ \beta_{r+1} \sum_{1}^{r} c_{jir+1}\beta_j \frac{\partial}{\partial \beta_{r+1}} \quad (i = 1, 2, \cdots, r). \end{aligned} \tag{30}$$

Then, if $f(\alpha) \equiv \mathit{ff}(\beta)$,

$$\overline{D}_i \mathit{ff}(\beta) = D_i f(\alpha) \qquad (i = 1, 2, \cdots, r). \tag{31}$$

Therefore,

$$\begin{aligned} \left(\sum_{1}^{r} c_{ji1}\beta_j - \beta_1 \sum_{1}^{r} c_{jir+1}\beta_j\right) \frac{\partial F(\beta)}{\partial \beta_1} + \cdots + \beta_{r+1} \sum_{1}^{r} c_{jir+1}\beta_j \frac{\partial F(\beta)}{\partial \beta_{r+1}} \\ \equiv \overline{D}_i F(\beta) \equiv D_i\psi(\alpha) \equiv 0 \quad (i = 1, 2, \cdots, r). \end{aligned} \tag{32}$$

It follows that $F(\beta)$ is a solution of the system of equations

$$\overline{D}_1 f(\beta) = 0, \cdots, \overline{D}_r f(\beta) = 0. \tag{33}$$

Since the coefficients of this system of equations are rational integral functions of the β's, the solution $F(\beta)$ of this system may be taken holomorphic in the neighborhood of $\beta_1 = 0, \cdots, \beta_r = 0, \beta_{r+1} = 1$. Consequently

$$\psi(\alpha) \equiv F\left(\frac{\alpha_1}{\alpha_{r+1}}, \cdots, \frac{\alpha_r}{\alpha_{r+1}}\right)$$

may be taken holomorphic in the neighborhood of $\alpha_1 = 0, \cdots, \alpha_r = 0, \alpha_{r+1} = 1$. There are two cases: first, $\psi(\alpha)$ is algebraic; second, $\psi(\alpha)$ is transcendental.

The Function ψ (α) Algebraic Invariant.—The invariant $\psi(\alpha)$ is now assumed to be algebraic. In this case the invariant spread $\psi(\alpha) = 0$ is represented by the rational integral equation of degree m homogeneous in the α's,

$$\Psi(\alpha) \equiv u_0(\alpha)\alpha_{r+1}^{m} + u_1(\alpha)\alpha_{r+1}^{m-1} + \cdots + u_k(\alpha)\alpha_{r+1}^{m-k} + \cdots + u_m(\alpha) = 0, \tag{34}$$

where $u_k(\alpha)$ is a polynomial of degree k homogeneous in $\alpha_1, \cdots, \alpha_r$.

Since the point $(0, \cdots, 0, 1)$, i.e., the point $\alpha_1 = \cdots = \alpha_r = 0, \alpha_{r+1} = 1$, corresponding to the exceptional infinitesimal transformation X_{r+1} of G, is invariant to the adjoint of G,* the polars of $(0, \cdots, 0, 1)$ quâ $\Psi(\alpha) = 0$ are all invariant to the adjoint of G. These polars are:

$$
(35)\quad
\begin{aligned}
\frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} &\equiv \frac{m!}{(m-1)!} u_0(\alpha)\alpha_{r+1}^{m-1} + \frac{(m-1)!}{(m-2)!} u_1(\alpha)\alpha_{r+1}^{m-2} \\
&\qquad + \cdots + \frac{1!}{0!} u_{m-1}(\alpha) = 0, \\
\frac{\partial^2 \Psi(\alpha)}{\partial \alpha_{r+1}^2} &\equiv \frac{m!}{(m-2)!} u_0(\alpha)\alpha_{r+1}^{m-2} + \frac{(m-1)!}{(m-3)!} u_1(\alpha)\alpha_{r+1}^{m-3} \\
&\qquad + \cdots + \frac{2!}{0!} u_{m-2}(\alpha) = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial^{m-2} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-2}} &\equiv \frac{m!}{2!} u_0(\alpha)\alpha_{r+1}^{2} + \frac{(m-1)!}{1!} u_1(\alpha)\alpha_{r+1} \\
&\qquad + \frac{(m-2)!}{0!} u_2(\alpha) = 0, \\
\frac{\partial^{m-1} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-1}} &\equiv \frac{m!}{1!} u_0(\alpha)\alpha_{r+1} + \frac{(m-1)!}{0!} u_1(\alpha) = 0.
\end{aligned}
$$

In the equation

$$\Psi(\alpha) \equiv u_0(\alpha)\alpha_{r+1}^{m} + u_1(\alpha)\alpha_{r+1}^{m-1} + \cdots + u_m(\alpha) = 0$$

we may either have $u_0(\alpha) \neq 0$, in which case the point $(0, \cdots, 0, 1)$ is not on the spread $\Psi(\alpha) = 0$, or $u_0(\alpha) = 0$, and thus the point $(0, \cdots, 0, 1)$ is on the spread $\Psi(\alpha) = 0$. In the former case, there is an $(r-1)$-flat not passing through the point $(0, \cdots, 0, 1)$, namely, the $(m-1)$th polar of that point quâ $\Psi(\alpha) = 0$,

$$(36)\qquad \frac{\partial^{m-1} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-1}} \equiv \frac{m!}{1!} u_0(\alpha)\alpha_{r+1} + \frac{(m-1)!}{0!} u_1(\alpha) = 0,$$

invariant to the adjoint of G.

Let $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(r)}$, where

$$\alpha^{(\nu)} \equiv (\alpha_1^{(\nu)}, \cdots, \alpha_r^{(\nu)}, \alpha_{r+1}^{\nu}) \qquad (\nu = 1, \cdots, r),$$

be any r points in the $(r-1)$-flat $\dfrac{\partial^{m-1}\Psi(\alpha)}{\partial \alpha_{r+1}^{m-1}} = 0$ not lying in any $(r-2)$-flat. These r points together with the point $\alpha^{(r+1)} \equiv (0, \cdots, 0, 1)$ then constitute a system of $r+1$ points not lying in any r-flat.

Let us transform the coördinate system by the transformation

$$(37)\qquad \alpha_i = \bar{\alpha}_1\alpha_i^{(1)} + \cdots + \bar{\alpha}_r\alpha_i^{(r)} + \bar{\alpha}_{r+1}\alpha_i^{(r+1)} \quad (i = 1, 2, \cdots, r, r+1),$$

* Lie-Scheffers, "Continuirliche Gruppen," pp. 465, 485.

where

$$\alpha_i^{(r+1)} = 0 \quad (i \neq r+1), \qquad \alpha_{r+1}^{(r+1)} = 1.$$

In Cayley's notation this transformation can be written:

$$(\alpha_1, \cdots, \alpha_r, \alpha_{r+1}) = \begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_1^{(r)}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{r+1}^{(1)}, & \cdots, & \alpha_{r+1}^{(r)}, & 0 \end{vmatrix} (\bar{\alpha}_1, \cdots, \bar{\alpha}_r, \bar{\alpha}_{r+1}).$$

And now if

$$\bar{X}_i = \alpha_i^{(1)} X_1 + \cdots + \alpha_r^{(i)} X_r + \alpha_{r+1}^{(i)} X_{r+1} \quad (i = 1, 2, \cdots, r, r+1), \tag{38}$$

or

$$(\bar{X}_1, \cdots, \bar{X}_r, \bar{X}_{r+1}) = \begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)}, & \alpha_{r+1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)}, & \alpha_{r+1}^{(r)} \\ 0, & \cdots, & 0, & 1 \end{vmatrix} (X_1, \cdots, X_r, X_{r+1}),$$

then

$$\sum_{i=1}^{r+1} \bar{\alpha}_i \bar{X}_i = \sum_{i=1}^{r+1} \alpha_i X_i. \tag{39}$$

The infinitesimal transformations $\bar{X}_1, \cdots, \bar{X}_r, \bar{X}_{r+1}$ will then be independent, since the determinant

$$\begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)}, & \alpha_{r+1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)}, & \alpha_{r+1}^{(r)} \\ 0, & \cdots, & 0, & 1 \end{vmatrix} = \begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)} \end{vmatrix} \neq 0;$$

and we shall have

$$(\bar{X}_i, \bar{X}_j) = \sum_{k=1}^{r+1} \bar{c}_{ijk} \bar{X}_k \quad (i, j = 1, 2, \cdots, r, r+1). \tag{40}$$

Since $\bar{X}_1, \cdots, \bar{X}_r$ are represented by the points $\alpha^{(1)}, \cdots, \alpha^{(r)}$ in the $(r-1)$-flat

$$\frac{(m-1)!}{1!} u_0(\alpha)\alpha_{r+1} + \frac{(m-1)!}{0!} u_1(\alpha) = 0$$

which is invariant to the adjoint of G, they will constitute an invariant subgroup of G;* and thus

$$(\bar{X}_i, \bar{X}_j) = \sum_{k=1}^{r} c_{ijk} \bar{X}_k \quad (i = 1, 2, \cdots, r; j = 1, \cdots, r, r+1). \tag{41}$$

* Lie-Scheffers, "Continuirliche Gruppen," pp. 485–487.

Further, $\overline{X}_{r+1} = X_{r+1}$. Whence

$$\bar{c}_{ijr+1} = - \bar{c}_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r+1), \tag{42}$$

which was to be proved. We may therefore assume that $u_0(\alpha) = 0$.

If now $m = 1$, the equation of the spread invariant to the adjoint of G is $\Psi(\alpha) \equiv u_1(\alpha) = 0$. But in this case $\dfrac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} \equiv 0$, which is contrary to our assumption.

Let $m = 2$. The equations of the invariant spread and the invariant polars are then

$$\Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} + u_2(\alpha) = 0, \qquad \frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} \equiv u_1(\alpha) = 0.$$

Let

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \not\equiv 0, \qquad u_2(\alpha) \not\equiv u_1(\alpha) \cdot v_1(\alpha),$$

where $v_1(\alpha)$ is a function linear in $\alpha_1, \cdots, \alpha_r$. If now

$$u_1(\alpha) = 0, \qquad \text{then} \qquad \frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} = 0;$$

and since this equation is invariant, $u_1(\alpha') = \dfrac{\partial \Psi(\alpha')}{\partial \alpha'_{r+1}} = 0$, where $\alpha'_1, \cdots, \alpha'_r$, α'_{r+1} are the coördinates of the point obtained by applying the infinitesimal transformations of the adjoint of G to the point $(\alpha_1, \cdots, \alpha_{r+1})$. Again, if

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0,$$

then

$$\frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} \equiv u_1(\alpha) = 0, \qquad \Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} + u_2(\alpha) = 0;$$

and, since these equations are separately invariant to the adjoint of G, it follows that

$$\frac{\partial \Psi(\alpha')}{\partial \alpha'_{r+1}} \equiv u_1(\alpha') = 0, \qquad \Psi(\alpha') \equiv u_1(\alpha')\alpha'_{r+1} + u_2(\alpha') = 0;$$

and thus

$$u_1(\alpha') = 0, \qquad u_2(\alpha') = 0.$$

Therefore, the spread

$$u_1(\alpha) = 0$$

and the spread

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0$$

are both invariant to the adjoint of G', which is contrary to the assumption that the adjoint of G' has but one invariant spread.

If

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \equiv 0,$$

then the invariant spread

$$\Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} = 0$$

is reducible. In this case each component of that spread is invariant to the adjoint of G; and therefore,

$$\alpha_{r+1} = 0$$

is an invariant flat not passing through the point $(0, \cdots, 0, 1)$, a case already treated above. In this case we can clearly see that

$$c_{ijr+1} = - c_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r + 1).$$

If

$$u_1(\alpha) \equiv 0, \qquad u_2(\alpha) \not\equiv 0,$$

then

$$\frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} \equiv 0,$$

which is contrary to the assumption.

If

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \equiv v_1(\alpha) \cdot u_1(\alpha) \not\equiv 0,$$

where $v_1(\alpha)$ denotes a function linear in $\alpha_1, \cdots, \alpha_r$, then the invariant spread

$$\Psi(\alpha) \equiv u_1(\alpha) \cdot \alpha_{r+1} + u_2(\alpha) \equiv u_1(\alpha)[\alpha_{r+1} + v_1(\alpha)] = 0$$

is reducible; and the component flat

$$\alpha_{r+1} + v_1(\alpha) = 0$$

is invariant and does not pass through the point $(0, \cdots, 0, 1)$, in which case, as was shown above, we can, by a suitable choice of $X_1, \cdots, X_r, X_{r+1}$, make

$$c_{ijr+1} = - c_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r + 1).$$

Let now m be any positive integer greater than 2. The equations of the invariant spread and the invariant polars are then

$$\Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1}^{m-1} + u_2(\alpha)\alpha_{r+1}^{m-2} + \cdots + u_m(\alpha) = 0,$$

$$\frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} \equiv \frac{(m-1)!}{(m-2)!} u_1(\alpha) \cdot \alpha_{r+1}^{m-2} + \cdots + \frac{1!}{0!} u_{m-1}(\alpha) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial^{m-2} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-2}} \equiv \frac{(m-1)!}{1!} u_1(\alpha)\alpha_{r+1} + \frac{(m-2)!}{0!} u_2(\alpha) = 0,$$

$$\frac{\partial^{m-1} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-1}} \equiv \frac{(m-1)!}{0!} u_1(\alpha) = 0.$$

First, let $u_1(\alpha) \not\equiv 0$. Then either $u_2(\alpha) \not\equiv 0$ does not contain $u_1(\alpha)$, or

$u_2(\alpha)$, $u_3(\alpha)$, $\cdots$, $u_p(\alpha)$ $(p \leqq m)$ each contain $u_1(\alpha)$. In the former case, as we have seen before, the spread

$$u_1(\alpha) = 0$$

and the spread

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0$$

are both invariant to the adjoint of G', which is contrary to our assumption. In the latter case

$$\begin{aligned} u_2(\alpha) &= v_1(\alpha) \cdot u_1(\alpha), \\ u_3(\alpha) &= v_2(\alpha) \cdot u_1(\alpha), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_{p-1}(\alpha) &= v_{p-2}(\alpha) \cdot u_1(\alpha), \\ u_p(\alpha) &= v_{p-1}(\alpha) \cdot u_1(\alpha), \end{aligned}$$

where $v_k(\alpha)$ is a function of degree k homogeneous in $\alpha_1, \cdots, \alpha_r$. In this case each of the invariant spreads

$$\frac{\partial^{m-q}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-q}} = 0 \qquad (q = 2, 3, \cdots, p)$$

is reducible. Therefore, the component flat

$$\frac{(m-1)!}{1!}\alpha_{r+1} + \frac{(m-2)!}{0!}v_1(\alpha) = 0$$

is invariant. This case has been treated above.

Secondly, let

$$u_1(\alpha) \equiv 0, \qquad u_2(\alpha) \not\equiv 0.$$

Either $u_3(\alpha) \not\equiv 0$ does not contain $u_2(\alpha)$, or $u_3(\alpha)$, $u_4(\alpha)$, $\cdots$, $u_p(\alpha)$ $(p \leqq m)$ each contain $u_2(\alpha)$. In the former case, by reasoning similar to that employed before, it appears that the spread

$$u_2(\alpha) = 0$$

and the spread

$$u_2(\alpha) = 0, \qquad u_3(\alpha) = 0$$

are both invariant to the adjoint of G'. In the latter case

$$\begin{aligned} u_3(\alpha) &= v_1(\alpha) \cdot u_2(\alpha), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_{p-1}(\alpha) &= v_{p-3}(\alpha) \cdot u_2(\alpha), \\ u_p(\alpha) &= v_{p-2}(\alpha) \cdot u_2(\alpha), \end{aligned}$$

where $v_k(\alpha)$ is a function of order k homogeneous in $\alpha_1, \cdots, \alpha_r$. In this case each of the invariant spreads

$$\frac{\partial^{m-q}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-q}} = 0 \qquad (q = 3, 4, \cdots, p)$$

is reducible. In particular the invariant spread

$$\frac{\partial^{m-3}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-3}} \equiv \frac{(m-2)!}{2!}u_2(\alpha)\cdot\alpha_{r+1} + \frac{(m-3)!}{1!}u_3(\alpha)$$
$$\equiv u_2(\alpha)\left[\frac{(m-2)!}{2!}\alpha_{r+1} + \frac{(m-3)!}{1!}v_1(\alpha)\right] = 0$$

is reducible. Therefore, the component flat

$$\frac{(m-2)!}{2!}\alpha_{r+1} + \frac{(m-3)!}{1!}v_1(\alpha) = 0$$

is invariant. This case has also been treated above.

By the same reasoning we may treat the other similar cases which arise.

The Function $\psi(\alpha)$ Transcendental Invariant.—Since the point $\alpha^{(1)} = (0, \cdots, 0, 1)$ is invariant to the adjoint of G, the spreads

$$\sum_{i=1}^{r+1}\alpha_i\left[\frac{\partial\psi(\alpha)}{\partial\alpha_i}\right]_{\alpha=\alpha^{(1)}} = 0,$$
$$\sum\sum_{i=1\,j=1}^{r+1}\alpha_i\alpha_j\left[\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j}\right]_{\alpha=\alpha^{(1)}} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\sum_{i_1=1}^{r+1}\cdots\sum_{i_\mu=1}^{r+1}\alpha_{i_1}\cdots\alpha_{i_\mu}\left[\frac{\partial^\mu\psi(\alpha)}{\partial\alpha_{i_1}\cdots\partial\alpha_{i_\mu}}\right]_{\alpha=\alpha^{(1)}} = 0,$$

etc. $\cdots\cdots\cdots\cdots\cdots$

are invariant to the adjoint of G. Wherefore, there is an algebraic spread invariant to the adjoint of G, a case already treated above, or all the partial differential coefficients of $\psi(\alpha)$ quâ $\alpha_1, \cdots, \alpha_{r+1}$ are zero for $\alpha = \alpha^{(1)}$. But this is impossible as is shown below.

We have seen that

$$\psi(\alpha) = F(\beta_1, \cdots, \beta_r),$$

where

$$\beta_1 = \frac{\alpha_1}{\alpha_{r+1}}, \qquad \cdots, \qquad \beta_r = \frac{\alpha_r}{\alpha_{r+1}}, \qquad \beta_{r+1} = \alpha_{r+1},$$

can be taken analytic in the neighborhood of the point $\beta_1^{(1)} = \cdots = \beta_r^{(1)} = 0$, $\beta_{r+1}^{(1)} = 1$; and, therefore,

$$\psi(\alpha) = F(\beta) = F(\beta^{(1)}) + \sum_{i=1}^{r}\beta_i\left[\frac{\partial F(\beta)}{\partial\beta_i}\right]_{\beta=\beta^{(1)}}$$
$$+\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\beta_i\beta_j\left[\frac{\partial^2F(\beta)}{\partial\beta_i\partial\beta_j}\right] + \cdots + \cdots.$$

Since

$$\frac{\partial\psi(\alpha)}{\partial\alpha_i} = \frac{\partial F(\beta)}{\partial\beta_i}\cdot\frac{1}{\alpha_{r+1}} \qquad (i = 1, 2, \cdots, r),$$
$$\frac{\partial\psi(\alpha)}{\partial\alpha_{r+1}} = -\frac{\alpha_1}{\alpha_{r+1}^2}\cdot\frac{\partial F(\beta)}{\partial\beta_1} - \cdots - \frac{\alpha_r}{\alpha_{r+1}^2}\cdot\frac{\partial F(\beta)}{\partial\beta_r},$$

we should have, if

$$\left[\frac{\partial\psi(\alpha)}{\partial\alpha_i}\right]_{\alpha=\alpha^{(1)}} = 0 \qquad (i = 1, 2, \cdots, r, r+1),$$
$$\left[\frac{\partial F(\beta)}{\partial\beta_i}\right]_{\beta=\beta^{(1)}} = 0 \qquad (i = 1, 2, \cdots, r).$$

Since

$$\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j} = \frac{1}{\alpha_{r+1}^2}\cdot\frac{\partial^2F(\beta)}{\partial\beta_i\partial\beta_j} \qquad (i = 1, 2, \cdots, r),$$
$$\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_{r+1}} = \frac{\partial^2\psi(\alpha)}{\partial\alpha_{r+1}\partial\alpha_i} = \frac{1}{\alpha_{r+1}^2}\cdot\frac{\partial^2F(\beta)}{\partial\beta_i\partial\beta_{r+1}} \qquad (i = 1, 2, \cdots, r+1),$$

we should have, if

$$\left[\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j}\right]_{\alpha=\alpha^{(1)}} = 0 \quad (i = 1, 2, \cdots, r+1),$$
$$\left[\frac{\partial^2F(\beta)}{\partial\beta_i\partial\beta_j}\right]_{\beta=\beta^{(1)}} = 0 \quad (i = 1, 2, \cdots, r), \text{ etc. } \cdots.$$

Thus in case the partial differential coefficients of $\psi(\alpha)$ quâ $\alpha_1, \cdots, \alpha_{r+1}$ are all zero for $\alpha = \alpha^{(1)}$, then

$$\psi(\alpha) \equiv F(\beta) \equiv 0,$$

which is contrary to the assumption.

Note.—Not in all the groups with $r+1$ essential parameters and one exceptional infinitesimal transformations can all the c_{ijr+1}'s $(i, j = 1, 2, \cdots, r)$ be made zero. The group with 5 essential parameters and one exceptional infinitesimal transformation, isomorphic (not holoedrically) with the integrable group of 4 essential parameters of the type,

$$(Y_1, Y_2) = 0, \qquad (Y_1, Y_3) = 0, \qquad (Y_2, Y_3) = Y_1,$$
$$(Y_1, Y_4) = Y_1, \qquad (Y_2, Y_4) = Y_2, \qquad (Y_3, Y_4) = 0,*$$

may serve as an illustration where not all the c_{ijr}'s $(i, j = 1, 2, 3, 4)$ can be made zero.

CLARK UNIVERSITY,
June 14, 1917.

* Lie-Scheffers, "Continuirliche Gruppen," p. 582.

THE LAPLACE-POISSON MIXED EQUATION.

By K. P. Williams.

In a recent number of this Journal * Borden discussed the Laplace-Poisson Mixed Equation

(1) $$f'(x+1) + p(x)f'(x) + q(x)f(x+1) + r(x)f(x) = 0.$$

He obtains two invariants that form a fundamental system for the equation. These invariants are

$$I(x) = \frac{r(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)},$$

$$J(x) = \frac{r(x)}{p(x)} - q(x-1).$$

If either invariant is zero, the equation takes a simple form. For instance, if $I(x) \equiv 0$, i.e., if

(2) $$r(x) \equiv p'(x) + p(x)q(x),$$

the equation (1) reduces to

(3) $$f(x+1) + p(x)f(x) \equiv Ce^{-\int q(x)dx},$$

where C is an arbitrary constant. This is a linear non-homogeneous difference equation of the first order. Borden obtains a solution of it by means of the symbolic solution $\Sigma G(x)$ of the equation

$$F(x+1) - F(x) = G(x),$$

where $G(x)$ is a known function.

Solutions obtained in this way are purely formal, and may have no real significance. Borden assumes at the outset of his paper that $p(x)$, $q(x)$, and $r(x)$ are analytic functions, but he nowhere makes an investigation to determine whether this hypothesis is sufficient to bestow any validity upon his results.

It is the purpose of this paper to investigate in certain cases the analytic character of the solutions Borden obtains. To do this it is necessary to make use of the existence theorems for linear difference equations. We shall state here the one that is sufficient for several cases that arise.

* R. F. Borden, "On the Laplace-Poisson Mixed Equation," American Journal of Mathematics, Vol. XLII, 1920, pp. 257–277.

THEOREM A.* *Consider the linear non-homogeneous difference equation*

$$g(x+1) + a(x)g(x) = b(x), \tag{4}$$

where $a(x)$ *and* $b(x)$ *are rational functions, so that*

$$a(x) = x^{\mu}\left(a_0 + \frac{a_1}{x} + \cdots\right),$$

$$b(x) = x^{\nu}\left(b_0 + \frac{b_1}{x} + \cdots\right), \qquad |x| > R.$$

There exists a series $g(x)$, *of the form*

$$\bar{g}(x) = x^{\nu-\mu}\left(g_0 + \frac{g_1}{x} + \cdots\right), \qquad \text{if} \qquad \mu > 0,$$

$$\bar{g}(x) = x^{\nu}\left(g_0 + \frac{g_1}{x} + \cdots\right), \qquad \text{if} \qquad \mu \leqq 0,$$

which formally satisfies (4), *but which in general diverges.*

There are two solutions $g_1(x)$ *and* $g_2(x)$ *with the following properties. The function* $g_1(x)$ *is analytic save at zeros of* $a(x)$, *poles of* $b(x)$, *and points congruent (mod 1) to these points on the left; and it is asymptotic to the formal series* $\bar{g}(x)$ *in the right half plane. The function* $g_2(x)$ *is analytic save for poles at the poles of* $a(x)$, $b(x)$, *and the points congruent to these points on the right; and it is asymptotic to* $\bar{g}(x)$ *in the left half plane.*†

From the relation (2) it is possible to determine any one of the three functions p, q, r in terms of the other two. We shall assume that p and q have the form of rational functions at infinity, so that

$$\begin{aligned} p(x) &= x^{m}\left(p_0 + \frac{p_1}{x} + \cdots\right), \\ q(x) &= x^{n}\left(q_0 + \frac{q_1}{x} + \cdots\right), \qquad |x| > R. \end{aligned} \tag{5}$$

There are four cases to consider, according as $n < -1$, $n = -1$, $n = 0$, $n > 0$.

* K. P. Williams, "The Solutions of Non-Homogeneous Linear Difference Equations and their Asymptotic Form," *Transactions of the American Math. Soc.*, Vol. XIV (1913), pp. 209–240.

† In order to be assured of solutions and to know their asymptotic forms, it is sufficient to assume that $a(x)$ and $b(x)$ have the form of rational functions at infinity. In that case, nothing can be said of the nature or situation of the singularities of the solutions in the finite part of the plane.

§ 1.

Case 1. Let $n < -1$; then we have

$$e^{-\int q(x)dx} = \left(1 + \frac{}{x^{-n-1}} + \frac{}{x^{-n-2}} + \cdots\right).^*$$

It is seen that equation (3) is now in the form (4), with $\mu = m$, $\nu = 0$. It follows from Theorem A that equation (3), and therefore equation (1), has two solutions, analytic in general, and these solutions are asymptotic in the right and left half planes, respectively, to a series

$$\bar{f}(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right), \qquad \text{if} \qquad m > 0,$$

$$\bar{f}(x) = f_0 + \frac{f_1}{x} + \cdots, \qquad \text{if} \qquad m \leqq 0,$$

which can be determined by direct substitution in (3).

The formal solution $\bar{f}(x)$ can, however, be directly obtained from (1). The calculation will be made only for the case $m > 0$.

When we substitute in (2) the values of $p(x)$ and $q(x)$ given in (5), we obtain

$$r(x) = x^{m-1}\left(mp_0 + \frac{}{x} + \cdots\right),$$

the highest power of x being the $(m-1)$th, since the assumption $n < -1$ gives $m - 1 > m + n$. The equation (1) then takes the form

$$f'(x+1) + x^m\left(p_0 + \frac{p_1}{x} + \cdots\right)f'(x)$$
$$+ x^n\left(q_0 + \frac{q_1}{x} + \cdots\right)f(x+1) + x^{m-1}\left(mp_0 + \frac{}{x} + \cdots\right)f(x) = 0.$$

Assume

$$f(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

and substitute, giving

$$-\frac{1}{x^{m+1}}(mf_0 + \cdots) - \frac{1}{x}(p_0 + \cdots)(mf_0 + \cdots)$$
$$+ \frac{1}{x^{m-n}}(q_0 + \cdots)(f_0 + \cdots) + \frac{1}{x}(mp_0 + \cdots)(f_0 + \cdots) = 0,$$

where we have omitted only powers of $1/x$. Remembering that $m > 0$, $n < -1$, we see that $f_0, f_1, f_2, \cdots$ can be determined step by step, the quantity f_0 being arbitrary.†

* We shall omit writing coefficients in series in $1/x$ where the coefficients can be determined in terms of given quantities, and their explicit form is of no concern.

† If $m \leqq 0$ it is found that f_0 is arbitrary, while some of the next coefficients, $f_1, f_2, \ldots$, are zero, the exact number having that value depending on m and n.

THEOREM 1. *Assume the coefficients p, q, and r satisfy the following hypotheses:*

(1) $$r(x) \equiv p'(x) + p(x)q(x).$$

(2) *$p(x)$ has a pole of order m at infinity ($m = 0$, if $p(x)$ is analytic at infinity).*

(3) *$q(x)$ is analytic at infinity, and has a zero there of at least the second order.*

Then there is a formal solution $\bar{f}(x)$ of equation (1) of the form

$$\bar{f}(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

which can be determined by direct substitution (f_0 is arbitrary). There are two solutions of (1), namely $f_1(x)$ and $f_2(x)$, analytic in general in the finite plane. Furthermore, $f_1(x)$ is asymptotic to $\bar{f}(x)$ in the right half plane, and $f_2(x)$ is asymptotic to $\bar{f}(x)$ in the left half plane.

§ 2.

Case 2. Let $n = -1$. In this case

$$e^{-\int q(x)dx} = x^{-q_0}\left(1 + \frac{}{x} + \frac{}{x^2} + \cdots\right).$$

The equation (3) is therefore of the form (4) with $\nu = -q_0$.*

By application of Theorem A we deduce

THEOREM 2. *Let p, q, and r satisfy hypotheses (1) and (2) of Theorem 1, and in addition suppose*

(3) *$q(x)$ is analytic at infinity, with a zero of the first order, and* $\lim_{x=\infty} xf(x) = q_0$.

Then there is a formal solution $\bar{f}(x)$ of (1) of the form

$$\bar{f}(x) = x^{-q_0-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

which can be determined by substitution. There are two solutions $f_1(x)$ and $f_2(x)$ with properties similar to those described in Theorem 1.

§ 3.

Case 3. Let $n = 0$. In this case

$$e^{-\int q(x)dx} = e^{-q_0x}x^{-q_1}\left(1 + \frac{}{x} + \cdots\right),$$

* It is not necessary that ν be an integer in (3). In fact the substitution

$$g(x) = x^{\nu}G(x)$$

will reduce (4) to a normal form with $\nu = 0$, and in which the expansion of $a(x)$ still begins with x^{μ}.

so that equation (3) is not of the form (4). It can, however, be reduced to that form by the substitution

$$f(x) = e^{-q_0 x} g(x).$$

The new equation will be

$$g(x+1) + \frac{1}{e^{q_0}} p(x) g(x) = \frac{x^{-q_1}}{e^{q_0}} \left(1 + \frac{\cdot}{x} + \cdots\right).$$

The application of Theorem A then gives

THEOREM 3. *Let p, q, r satisfy hypotheses* (1) *and* (2) *of Theorem* 1, *and in addition assume*

(3) *$q(x)$ is analytic at infinity, with*

$$\lim_{x=\infty} q(x) = q_0, \qquad \lim_{x=\infty} x(q(x) - q_0) = q_1.$$

Then there is a formal solution $\bar{f}(x)$ of (1) *of the form*

$$\bar{f}(x) = e^{-q_0 x} x^{-q_1 - m} \left(f_0 + \frac{f_1}{x} + \cdots\right).$$

Solutions $f_1(x)$ and $f_2(x)$ exist, and have properties similar to those described before.

§ 4.

Case 4. Let $n > 1$. We can now write

$$e^{-\int q(x)dx} = e^{-q^{(1)}(x)} \cdot q_2(x),$$

where

$$q^{(1)}(x) = q_0 \frac{x^{n+1}}{n+1} + \cdots + q_n x,$$

$$q^{(2)}(x) = x^{-q_{n+1}} \left(1 + \frac{q_{n+2}}{x} + \frac{\cdot}{x^2} + \cdots\right),$$

the quantities $q_0, q_1, \cdots$ having the significance given in (5). The equation (3) accordingly takes a form to which Theorem A is not applicable. A direct examination must therefore be made in order to determine whether a solution exists.

Consider the associated homogeneous equation

$$g(x+1) + p(x) g(x) = 0.$$

There is a formal solution

$$\bar{g}(x) = x^{mx} (-p_0 e^{-m})^x x^r \left(\bar{g}_0 + \frac{\bar{g}_1}{x} + \cdots\right), \tag{6}$$

*which in general diverges. There are two solutions $g_1(x)$ and $g_2(x)$, analytic in general in the finite plane, and asymptotic to $\bar{g}(x)$, in the right and left half planes, respectively.**

* G. D. Birkhoff, "General Theory of Linear Difference Equations," *Transactions of the American Math. Soc.*, Vol. 12 (1911), pp. 243–284.

Let $g(x)$ represent one (which one to be specified later) of these solutions, and put

$$f(x) = \omega(x) g(x), \tag{7}$$

We then have, upon substituting in (3), the equation

$$\omega(x+1) - \omega(x) = C \frac{e^{-q^{(1)}(x)} \cdot q^{(2)}(x)}{g(x+1)} = G(x). \tag{8}$$

The discussion of this equation falls into various cases.

Case 4a. Suppose $q_0 > 0$. Let $g(x)$ be $g_1(x)$. Then the series

$$\omega(x) = -G(x) - G(x+1) - \cdots \tag{9}$$

is uniformly convergent if the real part of x is positive and sufficiently large; and the series is a solution of (8). The proof in all its details will not be given, but is based upon the following considerations.* If s is sufficiently large, the sth term in (9) can be written, on account of the fact that $g_1(x)$ is asymptotic in the right half plane to $\bar{g}(x)$, ($g(\bar{x})$ having the form given in (6)),

$$\frac{e^{-q^{(1)}(x+s-1)}}{(x+s)^{m(x+s)}(-p_0 e^{-m})^{x+s}(x+s)^{r+q_{n+1}}} M(x, s),$$

where $M(x, s)$ is bounded. Now the dominant part of the numerator is, for large s,

$$e^{-\frac{q_0}{n+1}s^{n+1}}$$

and the dominant part of the denominator is $s^{ms+r+q_{n+1}} \cdot e^{-ms}$. The dominant part of the expression is therefore

$$e^{-\frac{q_0}{n+1}s^{n+1}+ms-(ms+r+q_{n+1})\log s}.$$

It is seen that this is the term of a rapidly convergent series, since q_0 is positive. This is true irrespective of the sign of m, for s^{n+1} dominates $s \log s$, for s large, if $n > 0$.

Case 4b. Suppose $q_0 < 0$, and let n be odd. Let $g(x)$ be $g_2(x)$. Then the series

$$\omega(x) = G(x-1) + G(x-2) + \cdots \tag{10}$$

is uniformly convergent for the real part of x negative and sufficiently large; and it is a solution of (8). The proof is effected as before, making use of the fact that $g_2(x)$ is asymptotic to $\bar{g}(x)$ in the left half plane.

Case 4c. Suppose $q_0 < 0$, and let n be even. Neither a series of form (9) nor (10) will converge in this case, with $g(x)$ representing either $g_1(x)$ or $g_2(x)$. This is true because s^{n+1} dominates $s \log s$.

A solution of (8) can be obtained by means of a contour integral due

* Williams, loc. cit., § 3.

to Guichard.* The integral

$$\omega(x) = \int \frac{G(t)dt}{e^{2\pi i(t-x)} - 1}$$

satisfies (8), when the path of integration passes between $x - 1$ and x, with the point x on the right, provided the path extends to infinity in such a way that the integral converges.

In order to determine a choice of the path of integration we shall examine the nature of

$$G(t) = \frac{Ce^{-q^{(1)}(t)} \cdot q^{(2)}(t)}{g(t+1)} = -\frac{Ce^{-q^{(1)}(t)} \cdot q^{(2)}(t)}{p(t)g(t)}.$$

It is obvious that we can neglect $q^{(2)}(t)$ and $p(t)$, if we change the exponent r in the asymptotic form of $\bar{g}(x)$ to $r' = r + m + q_{n+1}$. (This change has the effect of making $q^{(2)}(t)$ and $p(t)$ approach constants as $t = \infty$.)

Let $g(t)$ be $g_1(t)$; then the dominating part in the denominator of $|G(t)|$ is, for t large in the right half plane,

$$|t^{mt}(-p_0e^{-m})^t t^{r'}|.$$

If we put $t = u + iv$, and also $t = \tau e^{i\varphi}$ this becomes

$$e^{(mu+r')\log\tau-(m-\log-p_0)u-m\varphi v} = e^{(\tau m\cos\varphi+r')\log\tau-[(m-\log-p_0)\cos\varphi+m\varphi\sin\varphi]\tau}.$$

Now choose the path of integration so as to make it coincident with the lines

$$\varphi = \pm\frac{\pi}{n+1}$$

at a sufficient distance from $t = 0$. Then $t = \tau e^{i\frac{\pi}{n+1}}$ so that $t^{n+1} = -\tau^{n+1}$. It follows that the dominant part of the real portion of the polynomial $q^{(1)}(t)$ will be, for τ sufficiently great, the positive quantity

$$\frac{|q_0|\tau^{n+1}}{n+1}.$$

Along the contour considered it follows that

$$|G(t)| < M(\tau)e^{-\frac{|q_0|}{n+1}\tau^{n+1}-(\tau m\cos\varphi+r')\log\tau-[(m-\log-p_0)\cos\varphi+m\varphi\sin\varphi]\tau},$$

where $M(\tau)$ is bounded. It is seen that $G(t)$ will approach zero more rapidly than $e^{-\tau}$ as τ approaches infinity, and this irrespective of the value of m, r', and ρ.

A glance will reveal the behavior of the denominator in the Guichard integral on the distant part of the path of integration. Let $x = \xi + i\eta$.

* Williams, loc. cit., § 2.

Then

$$e^{2\pi i(t-x)} - 1 = e^{2\pi i(u-\xi)} \cdot e^{-2\pi(v-\eta)} - 1.$$

It is apparent that along the ray for which $\varphi = \pi/(n+1)$ the quantity just written approaches -1, while along the ray from which $\varphi = -\pi/(n+1)$ it becomes large.

It follows from the considerations given that the integral converges, and will thus furnish a solution of the equation (8).

We shall not inquire into the existence of other solutions or into the behavior of the solution in the infinite part of the plane.

THEOREM 4. *Let p, q, and r satisfy conditions* (1) *and* (2) *of Theorem* 1; *and in addition let $q(x)$ have a pole at infinity. Then it is possible to find a solution of* (8), *such that* (7) *will give a solution of equation* (3), *and therefore of equation* (1).

§ 5.

In the identity (2) we have thus far assumed that $p(x)$ and $q(x)$ were of given form at infinity.

Suppose that $r(x)$ and $p(x)$ are given; then

$$q(x) = \frac{r(x) - p'(x)}{p(x)}.$$

If $p(x)$ is analytic at infinity we may have any of the four cases, depending on the behavior of $r(x)$. If $p(x)$ has a pole at infinity, and $r(x)$ is analytic, or has a pole of lower order than $p(x)$, the equation reduces to Case 2, or possibly Case 1. If $r(x)$ has a pole of the same order as $p(x)$, we have Case 3. If $p(x)$ has a zero at infinity and $r(x)$ does not, the function $q(x)$ will have a pole, and we have Case 4. If $r(x)$ also has a zero at infinity we may have any of the four cases.

Suppose $q(x)$ and $r(x)$ are given. Then

$$p(x) = e^{-\int q(x)dx}\left[\int e^{\int q(x)dx} r(x)dx + C\right].$$

Assume that

$$q(x) = x^{n_1}\left(q_0 + \frac{q_1}{x} + \cdots\right),$$

$$r(x) = x^{n_2}\left(r_0 + \frac{r_1}{x} + \cdots\right), \qquad |x| > R.$$

In order that $p(x)$ have the form of a rational function at infinity we must assume that $n_1 < 0$. If $n_1 = -1$, we must in addition assume that q_0 is an integer (positive or negative) such that $q_0 + n_2 \neq -1$. In all cases not excluded equation (1) will come under a form already treated.

BLOOMINGTON, INDIANA,
March, 1921.

THE FOUR COLOR PROBLEM.*

By Philip Franklin.

1. By a map we shall understand a subdivision of an inversion plane or sphere by means of a finite number of circular arcs into a finite number of regions, which completely cover it. There is no loss of generality in this restriction, as a "map" on any surface of genus zero, with a finite number of regions bounded by simple curves, may be deformed into a map of the type just described. A *side* is a line along which two distinct regions touch each other; a *vertex* is a point which belongs to three or more regions. The problem of coloring a map with a given number of colors (denoted in what follows by A, B, C, etc.) is the association of a color with each region in such a way that any two regions with a side in common are given different colors. Two regions with only a vertex (or a finite number of vertices) in common may of course have the same color.

Whether every map can be colored with four colors is an outstanding question, for while no map has ever been exhibited which could not be colored with four colors, no rigorous demonstration of the possibility for the general case has ever been given.† It is known that four colors are necessary to color some maps and that five colors are sufficient to color all maps. If any maps which can not be colored in four colors exist, there must be one such map of a minimum number of regions. We will call such a map an *irreducible* map. It is known that an irreducible map has the following properties:‡

1. Each vertex belongs to three and only three regions.
2. No group of less than five regions forms a multiply-connected portion of the map. (Consequently there are no two-, three- or four-sided regions and no multiply-connected regions.)
3. No group of five regions forms a multiply-connected portion of the map unless the group consists of the five regions surrounding a pentagon.
4. No edge is surrounded by four pentagons.
5. No region is completely surrounded by pentagons.

* Presented to the National Academy of Sciences, November 17, 1920.

† A history of the question, with a bibliography, is given in the thesis of Alfred Errera "Du Coloriage des Cartes, etc.," Brussels, 1921.

‡ For an account of these reductions of the problem, which are due to A. B. Kempe and G. D. Birkhoff, see a paper by the latter "The Reducibility of Maps," American Journal of Mathematics, Vol. XXXV (1913), p. 116.

6. No even-sided region is completely surrounded by hexagons. Each of these statements amounts to saying that a certain configuration is not possible in an irreducible map. Such a configuration will be called a *reducible* configuration.

In this paper we shall derive a few additional reducible configurations, by means of which it can be shown that the number of regions in an irreducible map is greater than 25. An example is also given of a map of 42 regions which, although colorable in four colors, satisfies all the conditions which have been derived for an irreducible map. It may be taken as showing the extreme lack of generality of the results thus far obtained for this problem.

2. It is interesting to find out some of the properties of a map not containing any region of less than five sides, as this is a property of an irreducible map. Since our map is drawn on a sphere, the Euler formula (applied to a manifold of genus zero) gives:

$$a_0 - a_1 + a_2 = 2, \tag{1}$$

where a_0, a_1, a_2 are the number of vertices, sides and regions respectively. Also since only three regions touch any one vertex, we have:

$$2a_1 = 3a_0 = \sum_5 \nu A_\nu, \tag{2}$$

where A_ν means the number of regions of ν sides in the map; the last two expressions are each equal to the first since they represent twice the number of sides in the map, counted first with reference to vertices, then with reference to regions. From (1) and (2) we obtain:

$$a_1 = 3(a_2 - 2), \qquad a_0 = 2(a_2 - 2). \tag{3}$$

From (2), (3) and the fact that $a_2 = \sum_5 A_\nu$, we see that

$$6(\sum_5 A_\nu - 2) = \sum_5 \nu A_\nu. \tag{4}$$

This may be written:

$$A_5 = 12 + \sum_7 (\nu - 6) A_\nu \tag{5}$$

and since the second term on the right is positive, A_5 must be at least 12, and we have the well-known theorem:

*Every map containing no triangles or quadrilaterals and having three regions abutting on each vertex contains at least twelve pentagons.**

* Cf. Kempe, A. B., "The Geographical Problem of the Four Colors," AMERICAN JOURNAL OF MATHEMATICS, Vol. II (1879), p. 198.

We shall also prove that such a map must contain either:

A pentagon adjacent to two other pentagons,
A pentagon adjacent to a pentagon and to a hexagon, or
*A pentagon adjacent to two hexagons.**

For, consider a map with none of these combinations of regions and let us count the number of vertices in the map which belong to a hexagon or a pentagon. We find that the number of vertices contributed by hexagons nowhere in contact with pentagons will be greater than twice the number of such hexagons since each hexagon has six vertices and no vertex belongs to more than three hexagons; pentagons isolated from hexagons or other pentagons will give five vertices each; two pentagons adjacent to each other but to no other pentagons or hexagons give eight vertices together, and hence average four each; while a pentagon adjacent to a hexagon gives over four vertices, since of its five vertices we need only deduct two thirds to account for the two where the hexagon joins it. Thus if none of the three conditions enumerated above existed, the number of vertices would be at least $4A_5 + 2A_6$. That is we would have to have:

$$a_0 \geq 4A_5 + 2A_6. \tag{6}$$

But from (5) and the obvious inequality:

$$0 \geq \sum_7 (7 - \nu)A_\nu \tag{7}$$

there results:

$$\sum_7 A_\nu + 12 \leq A_5 \tag{8}$$

or

$$\sum_5 A_\nu + 12 \leq 2A_5 + A_6 \tag{9}$$

and since (from (3)):

$$\sum_5 A_\nu = a_2 = a_0/2 + 2, \tag{10}$$

$$a_0/2 + 14 \leq 2A_5 + A_6, \tag{11}$$

$$a_0 + 28 \leq 4A_5 + 2A_6, \tag{12}$$

which contradicts (6) and thus proves the theorem.

The above theorem is not restricted to irreducible maps, but it follows from it that if the configurations there shown to be present were reducible there could not be any irreducible maps and the four-color problem would be solved. While it does not appear to be possible to prove this, there are a number of more complicated configurations which are reducible.

* That every such map contains either two adjacent pentagons or a pentagon adjacent to a hexagon was proved by Wernicke, P., "Über den Kartographischen Vierfarbensatz, Mathematische Annalen," Vol. 58 (1904), p. 419.

3. To obtain these configurations, and prove their reducibility, we shall need the notion of *chains*, originated by Kempe,* and employed by Birkhoff.* If a map is colored, or partially colored, a group of regions colored in two colors (say A and B), forming a connected region, and such that each region adjacent to a region of the group is either colored in one of the remaining two colors (C or D) or not yet colored, is said to form a *chain* (an AB chain). Evidently we may obtain a second coloration or partial coloration of the map by interchanging the two colors on a single chain, and unless the map contains only one chain in this pair of colors, the new coloration will differ from the old by more than a mere permutation of the colors of the whole map.

Furthermore since two chains with no color in common, as an AB chain and a CD chain, can not "cross" each other, if we have a closed circuit consisting of an AB chain, or an AB chain and an uncolored region in the case of a partially colored map, it follows that the C and D regions on one side of the closed circuit can not belong to the same CD chain as those on the opposite side of the circuit. Thus in Fig. 1, if 1 is an uncolored region,

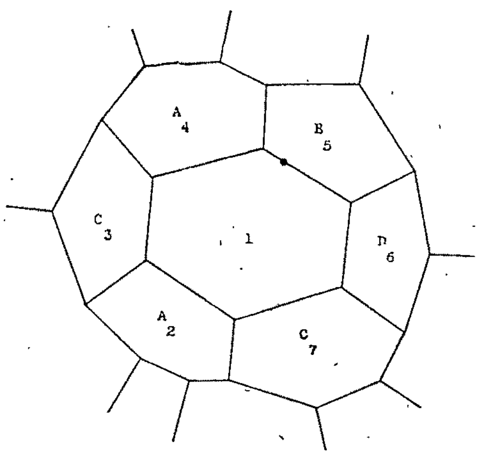

FIG. 1.

and 2 and 4 are joined by an AB chain, 3 must belong to a CD chain distinct from the one containing 6 and 7. Consequently we may interchange the colors in the CD chain containing 3 without affecting 6 and 7. Since, in most of the applications of this process, we shall only be concerned with the arrangement of the colors about the uncolored region, and the rest of them are unchanged by this operation, we shall briefly refer to this operation as "changing 3 to a D." The value of these operations will be seen in the proofs which follow.

* L. c.

4. We shall now prove that

A side of a hexagon surrounded by this hexagon and three pentagons is a reducible configuration. For, if it were present in an irreducible map, and we erased the dotted lines as in Fig. 2, we would obtain a new map which

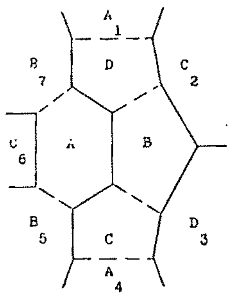

FIG. 2 *a*.

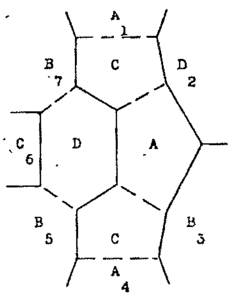

FIG. 2 *b*.

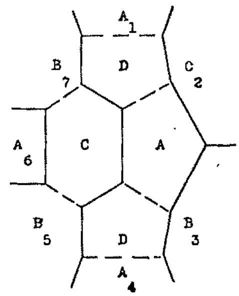

FIG. 2 *c*.

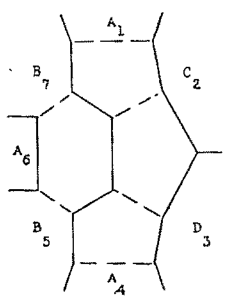

FIG. 2 *d*.

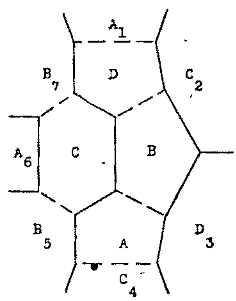

FIG. 2 *e*.

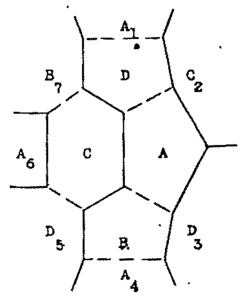

FIG. 2 *f*.

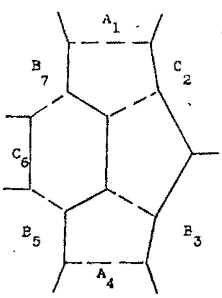

FIG. 2 *g*.

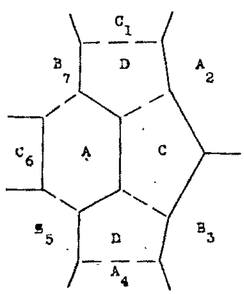

FIG. 2 *h*.

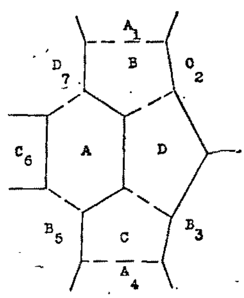

FIG. 2 *i*.

would contain fewer regions than an irreducible map and hence be colorable. From the way we selected the lines which were erased, regions 1 and 4 would have the same color (say A) while regions 5 and 7 would have a different common color (say B). Of the five essentially distinct colorations, the three cases shown in 2 *a*, 2 *b* and 2 *c* permit of immediate coloration, as indicated. In the case shown in 2 *d*, if 5 is joined to 7 by a BD chain, 6 may be changed to a C, reducing the problem to case 2 *a*, while if 5 is

joined to 3 by a BD chain, 4 may be changed to a C, and the map colored as shown in 2 *e*. If neither of these chains exist, 5 may be changed to a D, and the map colored as shown in 2 *f*. Finally in the case given in 2 *g*, either a BD chain joins 7 with 5, and we reduce to 2 *c* by changing 6 to A; or a BD chain joins 7 with 3, and we color as in 2 *h* after interchanging A and C in the AC chain including 1 and 2; or 7 may be changed to a D and we color as in 2 *i*.

If 5 and 7 had a side in common in our original map, we could not erase the dotted lines, and still leave a map; but in this case we would have three regions forming a multiply-connected piece. If 1 and 4 had a side in common, we would have a group of five regions forming a multiply-connected region of the map, and not all adjacent to the same pentagon. Hence both these cases are excluded by the properties of irreducible maps given in the first section.

If a pentagon is in contact with three pentagons, a hexagon, and a fifth region of any number of sides, either the hexagon is adjacent to this fifth region, in which case the three adjacent pentagons, with the initial pentagon, completely surround an edge, or the hexagon is adjacent to two pentagons, and we have a side of the hexagon completely surrounded by this hexagon and three pentagons. In either case, it is reducible and we have the result:

A pentagon in contact with three pentagons and a hexagon is a reducible configuration.

Also if a pentagon is in contact with two pentagons and three hexagons, if the two pentagons are not adjacent, they are separated by a hexagon, which with the three pentagons forms a reducible configuration. If the two pentagons are adjacent, we proceed as follows: We erase the boundaries which are dotted (Fig. 3 *a*) and color the resulting map. If all the regions

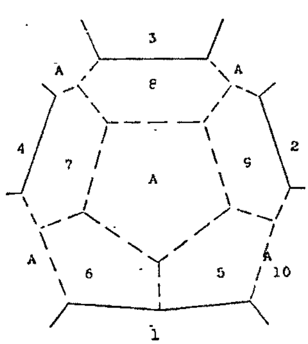

FIG. 3 *a*.

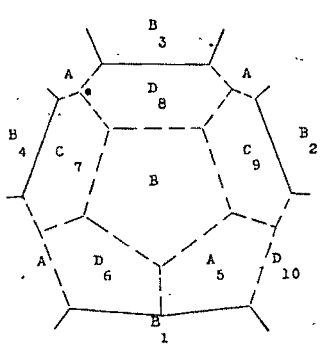

FIG. 3 *b*.

1, 2, 3, 4 are not colored in one color, there are two of them, say 1 and 2,

which are separated by a single region and in different colors, say B and C respectively. We then color 5 with C, and 6, 7, 8, 9 in turn, which is possible since taking them in this order we shall never come to a region surrounded by more than three different colors. On the other hand, if all the regions 1, 2, 3, 4 are colored in the same color, say B, either there is no BC chain joining all these regions, in which case we can change some of these regions to C and reduce our problem to the case just discussed, or the AD chain containing the region 10 is separated from the other regions marked A and can be changed to a D. The map is then colored as shown in Fig. 3 b. This proves the theorem:

A pentagon surrounded by two pentagons and three hexagons is a reducible configuration.

In this proof we have omitted any reference to the case where the dotted lines can not be erased without giving rise to a region which meets itself along one edge. We shall also do this in future cases where, as in this case, it may be excluded by the considerations used for this purpose in the proof of our first theorem.

By a method quite similar to the above, we could easily show that any odd-sided region, completely surrounded by one or more pairs of pentagons, the two of each pair being adjacent, and a number (necessarily odd) of hexagons, is a reducible configuration.

To lead up to a slightly more general theorem, we repeat Birkhoff's proof of the reducibility of an even-sided region surrounded by hexagons, for definiteness stating the proof for a hexagon so surrounded. We erase the dotted lines of Fig. 4 a and obtain the coloration shown. If all the

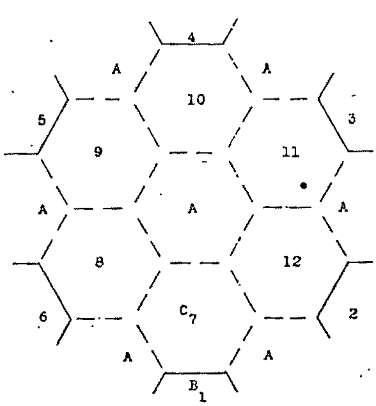

FIG. 4 a.

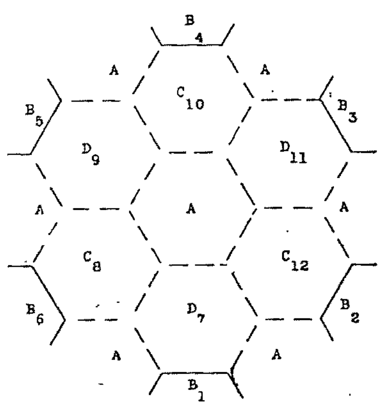

FIG. 4 b.

regions 1, 2, 3, 4, 5, 6 are not colored in one color, there are two of them, say 1 and 2, which are separated by a single region and in different colors,

say B and C. We then color 7 with C, and color 8, 9, 10, 11, 12 in turn, which is possible since we shall find each adjacent to regions of three different colors at most, and thus have a fourth with which to color each. If 1, 2, 3, 4, 5, 6 are all in the same color, our map is colored as in Fig. 4 *b*.

Our generalization is to the case where two adjacent hexagons are replaced by pentagons, and the above method is directly applicable, provided we imagine one of the regions marked A as shrunk to a point. This shows that:

An even-sided region completely surrounded by hexagons and pairs of pentagons, the two of each pair being adjacent, is a reducible configuration.

If a hexagon is surrounded by two hexagons and four pentagons, either the pentagons are grouped so as to come under the theorem just proved, or one of the edges of the hexagon is in contact with three pentagons which we proved above was a reducible configuration. Thus:

A hexagon surrounded by four pentagons and two hexagons is a reducible configuration.

A region of an even number of sides ($2n$) *surrounded by* $2n - 2$ *pentagons and two other regions, which are adjacent, is reducible.*

To fix the ideas, we state the proof for an octagon. We erase the dotted lines (Fig. 5) and color the resulting map. If 4 is a C, we color 15, 14, 13,

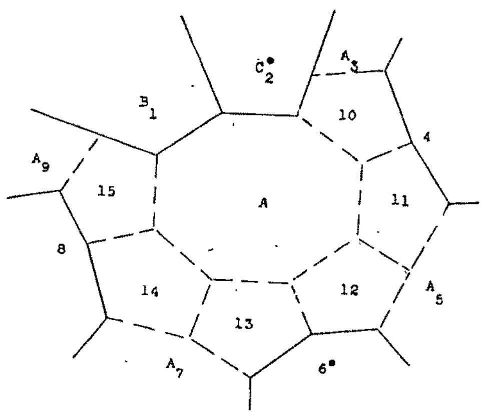

FIG. 5.

12, 11, 10 in turn which will be possible since each will only be adjacent to regions in at most three different colors when we come to it. If 4 is a B or D, we color 10 D or B respectively, and color 11 C. If 6 is a C, we color 15, 14, 13, 12 in turn, as before; while if it is B or D we color 12 D or B respectively and 13 C. We then color 14 and 15 D, C; B, D; or B, C; according as 8 is B, C or D.

A region of an odd number of sides ($2n - 1$) *surrounded by* $2n - 2$ *pentagons and one other region is reducible.*

We give the proof for a heptagon. After erasing the dotted lines (Fig. 6) we color the new map, as indicated. Reasoning exactly as for the

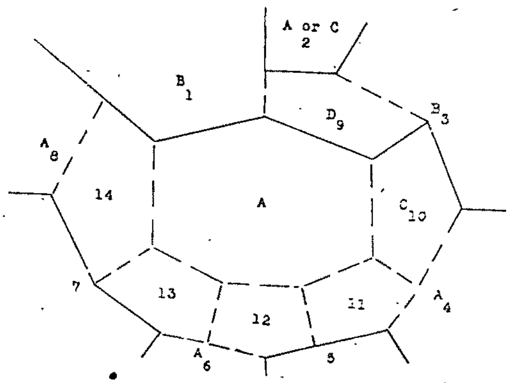

FIG. 6.

preceding theorem, we show that if 5 is a C the map is colorable, while if it is not C we color 11 and 12, giving 12 the color C. Then we color 13 and 14 D, C; B, D; or B, C; according as 7 is B, C or D.

It follows from our last two theorems that:

An n-gon in contact with n — 1 *pentagons is reducible.*

5. We will now deduce certain inequalities which must be satisfied by the numbers A_ν if the map is an irreducible one. We have shown that such a map must not contain:

(*a*) An n-gon in contact with $n - 1$ pentagons,
(*b*) A pentagon in contact with three pentagons and one hexagon,
(*c*) A pentagon in contact with two pentagons and three hexagons,
(*d*) A hexagon in contact with four pentagons and two hexagons,

in addition to the six configurations given in the first section, and have also shown that the equation (see (5)):

$$A_5 = 12 + \sum_7 (\nu - 6)A_\nu \tag{13}$$

applies to such a map.

From (*a*) we know that every region of our map is in contact with at least *two* regions of more than five sides. Hence the number of sides (a side being counted with each of the two regions it separates) of regions with more than five sides must be at least equal to twice the total number of regions. That is:

$$\sum_6 \nu A_\nu \geq 2 \sum_5 A_\nu. \tag{14}$$

To obtain a second inequality from the remaining conditions, we write:

$A_5^0 =$ the number of pentagons in contact with no region of more than six sides,
$A_5^1 =$ the number of pentagons in contact with only one such region,
$A_5^2 =$ the number of pentagons in contact with two or more such regions,
$A_6^0 =$ the number of hexagons in contact with no such regions, and
$A_6^1 =$ the number of hexagons in contact with at least one such region.

From (a) it follows that each region A_5^2, A_6^1 (we thus abbreviate regions of the type counted in A_5^2, A_6^1) as well as those of more than six sides is in contact with at least *two* regions of more than five sides. Also from (a), (b) and (c) it follows that each pentagon A_5^0 is in contact with at least *four* hexagons, and from (a) and (b) that each pentagon A_5^1 is in contact with at least two hexagons in addition to the one region of more than six sides; and therefore to at least *three* regions of more than five sides. Finally from (a) and (d) we see that each hexagon A_6^0 is in contact with at least *three* other hexagons. Thus we have:

$$(15)\qquad \sum_6 \nu A_\nu \geq 4A_5^0 + 3A_5^1 + 2A_5^2 + 3A_6^0 + 2A_6^1 + 2\sum_7 A_\nu.$$

But from the definitions of the regions A_5^1, etc.:

$$(16)\qquad \sum_7 \nu A_\nu \geq A_5^1 + 2A_5^2 + A_6^1.$$

If we add corresponding members of (15) and (16), recollecting that $A_5 = A_5^0 + A_5^1 + A_5^2$ and $A_6 = A_6^0 + A_6^1$, we obtain the result:

$$(17)\qquad \sum_6 \nu A_\nu + \sum_7 \nu A_\nu \geq 4A_5 + 3A_6 + 2\sum_7 A_\nu,$$

which may be written:

$$(18)\qquad 2\sum_7 \nu A_\nu \geq 4A_5 - 3A_6 + 2\sum_7 A_\nu.$$

For a map which contains no regions of more than seven sides, we may obtain a somewhat stronger inequality, by using:

$$(19)\qquad 4A_6 + 5A_7 \geq 4A_5^0 + 3A_5^1 + 2A_5^2,$$
$$(20)\qquad 5A_7 \geq A_5^1 + 2A_5^2,$$

which are analogous to (15) and (16); except that by considering only sides of pentagons in contact with regions of more than five sides we are enabled to use (a) in deriving the left members. They give:

$$(21)\qquad 4A_6 + 10A_7 \geq 4A_5,$$

which is only applicable to maps composed entirely of pentagons, hexagons, and heptagons.

By using (13) we may reduce (14) and (18) to the respective forms:

(22) $$\sum_{6} (10 - \nu)A_\nu \geq 24,$$

(23) $$2 \sum_{7} (11 - \nu)A_\nu \geq 48 - 3A_6.$$

These two inequalities show that the map we are considering must have at least 25 regions, and if it have only 25, they must be 17 pentagons, 3 hexagons, and 5 heptagons. For if the map contained two hexagons (22) would give:

(24) $$\sum_{7} (10 - \nu)A_\nu \geq 24 - 4A_6 = 16,$$

which requires at least six regions of more than six sides, and hence by (13) at least 18 pentagons. This would make 26 regions. If the map contained less than two hexagons, the same equation would show that there were more than 27 regions in the map, by a similar argument. Also our map can not have less than five regions of more than six sides, since if the map contained four heptagons, (23) would give:

(25) $$3A_6 > 16,$$

and the six hexagons required by (25) together with the 16 pentagons required by (13) would make 26 regions. Furthermore each heptagon less than four will increase the right member of (25) by eight, and hence require at least two additional hexagons in place of the heptagon and pentagon removed. Replacing any of the heptagons by octagons or regions of more than eight sides will strengthen our inequalities, as well as necessitating more pentagons to satisfy (13).

But the map of 17 pentagons, three hexagons, and five heptagons is not irreducible, since it does not satisfy (21). Consequently every irreducible map must contain more than 25 regions and this gives the theorem:

Every map containing 25 *or fewer regions can be colored in four colors.*

6. The question naturally arises whether 25 is the greatest number for which we can prove such a theorem as the above on the basis of the reductions already described. While an exact answer to this question is lacking, it is evident that the smallest number of regions in a map not containing any of these known reducible configurations is not considerably above 25, as we can construct a map with a small number of regions not containing any of them. Thus in Fig. 7 we exhibit a map of 42 regions which satisfies all the properties of irreducible maps given by previous writers as well as those derived in this paper. The map may be formed by constructing a hexagon on each of the 30 edges of a regular dodecahedron, in such a way

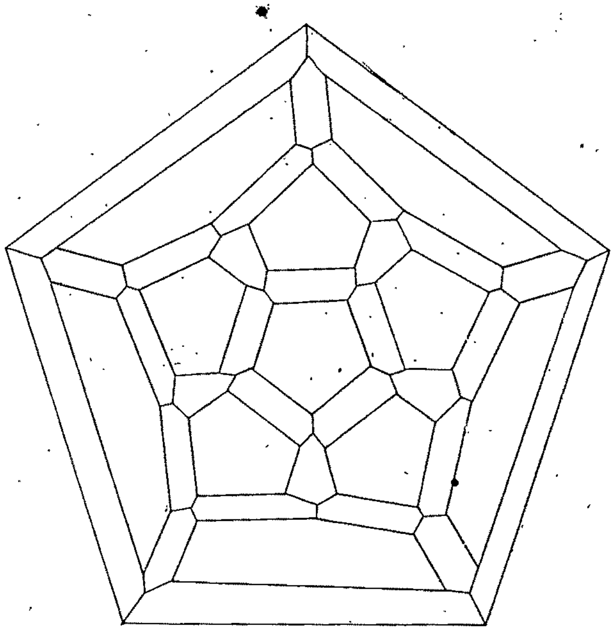

FIG. 7.

as to leave twelve pentagonal faces,* and may be colored by first coloring the pentagons as they would be colored for the dodecahedron.

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* This process was suggested to the writer in another connection by Prof. J. W. Alexander.

ON THE KERNEL OF THE STIELTJES INTEGRAL CORRESPONDING TO A COMPLETELY CONTINUOUS TRANSFORMATION.*

By Charles Albert Fischer.

A large part of the Fredholm theory of integral equations has been derived for the equation

(1) $$g(x) = f(x) - \lambda A(f),$$

where A is a completely continuous, linear transformation.† It is well known that every linear transformation can be put into the form

(2) $$A(f) = \int_a^b f(y) d_y K(x, y),$$

and the conditions which K must satisfy in order that A shall be completely continuous have been found.‡ The present paper discusses the relation between $K(x, y)$ and the solutions of the homogeneous equation corresponding to (1). In the first section some properties of orthogonal transformations are discussed, and it is proved that the limit of a uniformly convergent sequence of completely continuous, linear transformations is of the same type. In § 2 it is proved that a null element with respect to the transformation

(3) $$B_\lambda(f) = f(x) - \lambda A(f)$$

is a kernel element with respect to all other values of λ. Riesz has proved that A can be decomposed into the sum of two transformations A' and A'', such that the transformation $B'(f) = f(x) - A'(f)$ has a unique inverse, and the equations $B^n(f) = 0$ and $B''^n(f) = 0$ have the same solutions.§ In § 3 of the present paper it is proved that the $K''(x, y)$ corresponding to A'' can be put into the form

(4) $$K''(x, y) = \sum_{i=1}^{n} \varphi_i(x)\psi_i(y),$$

where $\varphi_1, \varphi_2, \cdots, \varphi_n$ are null elements. In the next section this theory is applied to the transformation B_λ, and in the last section the Stieltjes integral equation of the first kind is discussed.

§ 1. Orthogonal Transformations and Uniformly Convergent Sequences.

* Presented to the American Mathematical Society, April 23, 1921.

† F. Riesz, *Acta Mathematica*, Vol. 41 (1918), pp. 71–98.

‡ Fischer, *Bulletin of the American Mathematical Society*, Vol. 27 (1920), pp. 10–17.

§ Riesz, loc. cit., theorems 10 and 11.

—In all that follows $A(f)$, $A'(f)$, etc., will represent completely continuous, linear transformations carrying functions of the class $\{f\}$ into other functions of the same class. The class $\{f\}$ will be composed of all bounded functions defined on the interval (a, b), which are summable with respect to a function of bounded variation by Young's method of monotone sequences,* beginning with continuous functions. The transformations B, B', etc., will be defined by equations such as $B = E - A$, etc., where E is the identical transformation, and B_λ by equation (3). When $A(f)$ is put into the form (2), the function $K(x, y)$ is determined uniquely when it is required to satisfy the equations $K(x, a) = 0$, and $K(x, y) = K(x, y + 0)$, $(a < y < b)$.† If $K(x, y)$, considered as a function of x, belongs to $\{f\}$ for every value of y, and $f(y)$ is continuous, the integral (2) can be approached by a bounded sequence of functions such as

$$\sum_{i=1}^{n} f(\eta_i)[K(x, y_i) - K(x, y_{i-1})], \qquad (y_{i-1} < \eta_i < y_i),$$

each of which belongs to $\{f(x)\}$, and if f is a discontinuous function belonging to $\{f\}$ the integral (2) can be approached by a system of bounded sequences beginning with such integrals for continuous f's. Consequently the integral (2) must also belong to $\{f\}$.‡ If $K(x, y)$ does not belong to $\{f\}$ for some value of y, the function $f(y)$ can be defined in such a way that $A(f)$ will not belong to $\{f\}$.

The necessary and sufficient condition that $A(f)$ be completely continuous is that $V_y K(x, y)$, that is the variation of K considered as a function of y, shall be bounded uniformly, and that when $x_1, x_2, \cdots$ are chosen in such a way that $K(x_r, y)$ approaches a unique limit when r becomes infinite, the equation

$$\operatorname*{limit}_{r \to \infty} V_y[K(x_r, y) - \operatorname*{limit}_{r \to \infty} K(x_r, y)] = 0$$

shall be satisfied.§

The transformations A_1 and A_2 are said to be orthogonal if the equations $A_1A_2(f) = A_2A_1(f) = 0$ are satisfied identically in the argument f.

If A_1 and A_2 are orthogonal and $B_1(f) = 0$, then $A_2(f) = A_2A_1(f) = 0$, and if $A = A_1 + A_2$, the equation $B(f) = 0$ must also be satisfied. Under the same circumstances, it follows from definition that $B_1 + B_2 = E + B$ and that $B_1B_2 = B$. Then if $B(f) = 0$, $B_1(f) + B_2(f) = f$, and $B_1B_2(f) = B_2B_1(f) = 0$. Thus putting $f_1 = B_2(f)$ and $f_2 = B_1(f)$, the following theorem has been proved; if A_1 and A_2 are orthogonal, and $A_1 + A_2 = A$,

* Young, *Proceedings of the London Mathematical Society*, Vol. 13 (1914), p. 109.

† Fischer, *Annals of Mathematics*, Vol. 19 (1917), pp. 39–40.

‡ Daniell, *Annals of Mathematics*, Vol. 19 (1918), p. 290, theorem 7 (7).

§ Fischer, *Bulletin*, loc. cit., p. 14.

the necessary and sufficient condition that $B(f) = 0$ is that it be the sum of two functions, f_1 and f_2, such that $B_1(f_1) = B_2(f_2) = 0$. This can easily be generalized. If A_1 and A_2 are orthogonal, the equation $A^r(f) = A_1^r(f) + A_2^r(f)$ must be satisfied for $r = 1, 2, \cdots$, and consequently $B_1^n + B_2^n = E + B^n$ and $B_1^n B_2^n = B^n$. Consequently, the necessary and sufficient condition that $B^n(f) = 0$ is that f be the sum of an f_1 and an f_2 such that $B_1^n(f_1) = B_2^n(f_2) = 0$.

A sequence of transformations $A_1, A_2, \cdots$ will be said to converge uniformly to a transformation A, if for every $\epsilon > 0$ there is an n_ϵ independent of f and x, such that for $n \geqq n_\epsilon$ the inequality $|A_n(f) - A(f)| \leqq \epsilon \|f\|$ shall be satisfied. The notation $\|f\|$ represents the least upper bound of $|f(x)|$.

It will now be proved that the limit of a uniformly convergent sequence of completely continuous, linear transformations is completely continuous and linear. The limit of such a sequence is evidently distributive, and if it can be proved to be bounded it must be linear. If the transformations are put into the form

$$A_n(f) = \int_a^b f(y) d_y K_n(x, y),$$

$V_y K_n(x, y)$ is the least upper bound of $|A_n(f)|/\|f\|$. It follows from the definition of uniform convergence that $V_y[K_n(x, y) - K(x, y)]$ approaches zero uniformly in x as n becomes infinite. If the least upper bound of $V_y K_n(x, y)$ is then called M_n, and n is taken as large as the n_ϵ mentioned above, the inequality $|A(f)| \leqq (M_n + \epsilon)\|f\|$ must be satisfied. Therefore the transformation A is bounded and linear. If it were not completely continuous there would be a sequence $x_1, x_2, \cdots$ and an $\epsilon > 0$ such that $K(x_r, y)$ would approach a limiting function $k(y)$ as r became infinite, while

$$(5) \qquad V_y(K(x_r, y) - k(y)) > \epsilon, \qquad (r = 1, 2, \cdots).$$

The function $K_1(x_r, y)$ would have to converge to a function, which will be called $k_1(y)$, for a subset of these x's.* In the same way a subset of this subset would make $K_2(x, y)$ converge. In this way the sequences $x_1^{(n)}, x_2^{(n)}, \cdots$ could be determined in such a way that each is a subset of the preceding, and $K_n(x_r^{(n)}, y)$ would converge to a $k_n(y)$ as r became infinite. Consequently the one sequence $x_1^{(1)}, x_2^{(2)}, \cdots$ would make $K_n(x_r^{(r)}, y)$ converge for every n. If n were then taken large enough the inequality

$$(6) \qquad V_y(K_n(x, y) - K(x, y)) < \frac{\epsilon}{4}, \qquad (a \leqq x \leqq b),$$

would have to be satisfied, and then, since A_n is completely continuous, r could be taken so large that the inequalities

* Fischer, *Bulletin*, loc. cit., p. 13. This also has been proved by Helly.

$$(7) \qquad V_y(K_n(x_r^{(r)}, y) - K_n(x_{r+i}^{(r+i)}, y)) < \frac{\epsilon}{2}, \qquad (i = 1, 2, \cdots),$$

would be satisfied. Inequalities (6) and (7) would imply that

$$V_y(K(x_r^{(r)}, y) - K(x_{r+i}^{(r+i)}, y)) < \epsilon, \qquad (i = 1, 2, \cdots),$$

and since the variation of the limit of a sequence of functions cannot be greater than the limit of their variations,* inequality (5) could not be satisfied. That is $A(f)$ must be completely continuous.

The following example illustrates the fact that the limit of a non-uniformly convergent sequence of completely continuous transformations need not be completely continuous. The functions $K_1(x, y)$, $K_2(x, y)$, $\cdots$ corresponding to A_1, A_2, $\cdots$ will be defined as identically zero for all values of x excepting $x_r = 1 - 1/r$, $(r = 1, 2, \cdots)$, and for these values of x by the equations

$$\begin{aligned} K_n(x_r, y) &= 0, && (0 \leqq y < x_r), \\ K_n(x_r, y) &= 1, && (r = 1, 2, \cdots, n; \quad x_r \leqq y \leqq 1), \\ K_n(x_r, y) &= n/r, && (r = n, n+1, \cdots; x_r \leqq y \leqq 1). \end{aligned}$$

Each of these functions satisfies the conditions for complete continuity, and as n becomes infinite the function $K(x, y)$ corresponding to the limiting transformation satisfies the equations

$$\begin{aligned} K(x_r, y) &= 0, && (0 \leqq y < x_r), \\ K(x_r, y) &= 1, && (x_r \leqq y \leqq 1). \end{aligned}$$

Consequently

$$V_y[K(x_r, y) - \operatorname*{limit}_{r \to \infty} K(x_r, y)] = 2,$$

and $A(f)$ cannot be completely continuous.

It will now be proved that if the corresponding terms of two uniformly convergent sequences of completely continuous transformations are orthogonal, their limits are orthogonal. The sequences will be called A_1, A_2, $\cdots$ and $\overline{A}_1$, $\overline{A}_2$, $\cdots$. Since $A_n\overline{A}_n(f) = 0$, the inequality

$$\| A\overline{A}(f) \| \leqq \| A[\overline{A}(f) - \overline{A}_n(f)] \| + \| [A - A_n]\overline{A}_n(f) \|,$$

must be satisfied. The first term of the right member of this inequality must approach zero uniformly for a bounded set of f's, as n becomes infinite, because A is bounded and $\overline{A}_n$ approaches $\overline{A}$ uniformly, and the second term approaches zero for a similar reason. Therefore A and $\overline{A}$ must be orthogonal.

§ 2. **On Kernel Elements and Null Elements.**—An element $f(x)$ is said to be null with respect to A if it is a solution of the equation $B^\nu(f) = 0$, and

* Fischer, *Bulletin*, loc. cit., p. 13.

is said to be kernel if there is a g in $\{f\}$ which satisfies the equation $B^{\nu}(g) = f$. The integer ν is the smallest integer such that every solution of $B^{\nu+1}(f) = 0$ is also a solution of $B^{\nu}(f) = 0$.*

One of the theorems in § 1 might have been stated: If A_1 and A_2 are orthogonal and $A_1 + A_2 = A$, every element which is null with respect to either A_1 or A_2 is also null with respect to A, and every element which is null with respect to A is the sum of two elements one of which is null with respect to each of the transformations A_1 and A_2. Of course one of the two may be identically zero.

It will now be proved that if f is null with respect to $\lambda_1 A$, it must be kernel with respect to λA when $\lambda \neq \lambda_1$. To accomplish this it will first be proved that if $B_{\lambda_1}(f) = 0, f$ must be kernel with respect to λ, and second that if g is kernel with respect to λ and $B_{\lambda_1}(f) = g, f$ must also be kernel. Then since when $B^2_{\lambda_1}(f) = 0$, $g = B_{\lambda_1}(f)$ must be a solution of $B_{\lambda_1}(g) = 0$, every solution of $B^2_{\lambda_1}(f) = 0$, and in the same way every solution of $B^{\nu}_{\lambda_1}(f) = 0$, must be kernel with respect to λA. If $B(f_{\lambda_1}) = 0$, it follows from definition that $A^n(f) = f/\lambda_1^n$, and consequently

$$B^{\nu}_{\lambda}(f) = \left(1 - \frac{\lambda}{\lambda_1}\right)^{\nu} \cdot f.$$

That is $(1 - \lambda/\lambda_1)^{\nu} f$, and consequently f itself, is kernel with respect to λA. To prove the second part, it will be assumed that $B_{\lambda_1}(f) = g$, and g is kernel with respect to λA. Then by definition $\lambda_1 A[f] = f - g$, and

$$A^n[f] = \frac{1}{\lambda_1^n}\{f - g - \lambda_1 A(g) - \cdots - \lambda_1^{n-1} A^{n-1}(g)\}.$$

This makes it possible to put $B^{\nu}_{\lambda}(f)$ into the form

$$B^{\nu}_{\lambda}(f) = \left(1 - \frac{\lambda}{\lambda_1}\right)^{\nu} f + c_0 g + c_1 A(g) + \cdots + c_{\nu-1} A^{\nu-1}(g),$$

where c_0, c_1, $\cdots$ are finite constants determined by λ and λ_1. Since the left member of this equation, and every term of the right member except the first, is kernel with respect to λA, that term, and consequently f itself, must be kernel.

It has been proved† that if λ is a critical value, that is a value such that there are null elements with respect to λA, each f determines f' and f'' uniquely, such that $f = f' + f''$ and f' is kernel and f'' null with respect to λA. All of the critical values can be arranged in a sequence $\lambda_1, \lambda_2, \cdots$,‡ and f can then be decomposed into

$$f = f_1'' + f_2'' + \cdots + f_n'' + f_n', \qquad (n = 1, 2, \cdots),$$

* See Riesz, loc. cit., theorem 2.

† Riesz, loc. cit., theorem 8.

‡ Riesz, loc. cit., theorem 12.

where f'_n is kernel with respect to $\lambda_1, \lambda_2, \cdots, \lambda_n$, and f''_n is null with respect to λ_n and kernel with respect to all other values of λ.

If the series $f'' = f''_1 + f''_2 + \cdots$ converges uniformly, the function $f - f''$ is the limit of a uniformly convergent sequence of functions $f'_{n+1}, f'_{n+2}, \cdots$, each of which is kernel with respect to λ_n. Consequently it is kernel with respect to λ_n, $(n = 1, 2, \cdots)$.*

§ 3, **The Kernel** $K''(x, y)$.—It is proved in Riesz' theorem 10 that the transformation A determines the orthogonal transformations A' and A'', such that if f' is kernel and f'' null with respect to A, the equations

$$A''(f') = A'(f'') = 0, \qquad A'(f') = A(f'), \qquad A''(f'') = A(f''),$$

are all satisfied. His theorem 1′ states that all the kernel elements can be expressed linearly in terms of a finite number of them. These can be selected in the following way. The first ones $\varphi_1, \varphi_2, \cdots, \varphi_s$ will be a complete set of linearly independent solutions of the equation $B(f) = 0$. Then if $\nu > 1$ there must be one or more independent solutions of $B^2(f) = 0$, which are not solutions of $B(f) = 0$, and these will be called $\varphi_{s+1}, \varphi_{s+2}, \cdots, \varphi_{s+i}$. The solutions of $B^3(f) = 0$ will follow, etc., until all the linear independent null elements are exhausted. Since when $B^n(f) = 0$, $B(f)$ is a solution of $B^{n-1}(f) = 0$, $B(\varphi_i)$ must be a linear combination of $\varphi_1, \varphi_2, \cdots, \varphi_{i-1}$. The equations

$$B(\varphi_i) = \sum_{j=1}^{i-1} a_{ij}\varphi_j, \qquad (i = s + 1, s + 2, \cdots, r), \tag{8}$$

must then be satisfied, where the a_{ij}'s are constants. The φ's will also be determined so as to satisfy the equations $\| \varphi_i \| = 1$, $(i = 1, 2, \cdots, r)$. Every element f can be put into the form

$$f(x) = f'(x) + c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_r\varphi_r(x), \tag{9}$$

in one and only one way, where f' is a kernel element, and the c's are constants. It follows from the proof of Riesz' theorem 9 that there is a constant C, independent of f and x, such that

$$\left\| \sum_{i=1}^{r} c_i\varphi_i \right\| \leqq C \| f \|,$$

and since $\varphi_1, \varphi_2, \cdots, \varphi_r$ are linearly independent, there is an M, independent of f and x, which satisfies the inequalities

$$| c_i | = \| c_i\varphi_i \| \leqq M \left\| \sum_{i=1}^{r} c_i\varphi_i \right\| .\dagger$$

* It can be proved by means of Riesz' theorem 4 that the limit of a uniformly convergent set of kernel elements is kernel.

† See the proof of Riesz' lemma 4.

Consequently $|c_i| = MC\,||f||$. The transformation A'' can be decomposed into the sum of r transformations defined by the equations

$$A_i(f) = c_i A(\varphi_i), \qquad (i = 1, 2, \cdots, r), \tag{10}$$

where the c's are given by equation (9). These transformations are distributive and satisfy the inequalities $||A_i(f)|| \leqq MCM_A\,||f||$, where M_A is the least upper bound of $V_y K(x, y)$. Therefore they are linear, and since each must transform a bounded set of functions into a compact set, they are completely continuous.

The transformations A', A'' and A_i can be expressed by the equations

$$A'(f) = \int_a^b f(y) d_y K'(x, y),$$
$$A''(f) = \int_a^b f(y) d_y K''(x, y),$$
$$A_i(f) = \int_a^b f(y) d_y K_i(x, y), \qquad (i = 1, 2, \cdots, r),$$

where the K's are determined uniquely by the conditions given in § 1. Then it follows from definition that

$$K''(x, y) = \sum_{i=1}^{r} K_i(x, y). \tag{11}$$

Since $\varphi_1, \varphi_2, \cdots, \varphi_s$ are solutions of $B(f) = 0$, the equations

$$\int_a^b \varphi_i(y) d_y K_i(x, y) = \varphi_i(x), \qquad (i = 1, 2, \cdots, s), \tag{12}$$

must be satisfied, and equations (8) are equivalent to

$$\int_a^b \varphi_i(y) d_y K_i(x, y) = \varphi_i(x) - \sum_{j=1}^{i-1} a_{ij}\varphi_j(x), \qquad (i = s + 1, \cdots, r). \tag{13}$$

For convenience φ_i^* will be defined as φ_i if $i \leqq s$, and as $\varphi_i - \sum_{j=1}^{s-1} a_{ij}\varphi_{ij}$ if $i > s$. This reduces equations (12) and (13) to the one form

$$\int_a^b \varphi_i(y) d_y K_i(x, y) = \varphi_i^*(x), \qquad (i = 1, 2, \cdots, r). \tag{14}$$

The equations

$$\int_a^b \varphi_i(y) d_y K_j(x, y) = 0, \qquad (i \neq j), \tag{15}$$

also follow from the definition of A_i, and in general

$$\int_a^b f(y) d_y K_i(x, y) = c_i \varphi_i^*(x). \tag{16}$$

Consequently this integral must vanish identically in f for all values of x for which $\varphi_i^*(x) = 0$, and therefore $K_i(x, y)$ must vanish identically in y for all such values of x. The function $\psi_i(x, y)$ is then determined uniquely by the equation $K_i(x, y) = \varphi_i^*(x)\psi_i(x, y)$, excepting for values of x for which both members vanish, and for such values it can be defined arbitrarily. Equation (16) then becomes equivalent to

$$\int_a^b f(y) d_y \psi_i(x, y) = c_i.$$

It follows that if $\varphi_i^*(x_1) \neq 0$, the equation $\psi_i(x, y) = \psi_i(x_1, y)$ must be satisfied for all values of x for which $\varphi_i^*(x) \neq 0$, and $\psi_i(x, y)$ can be defined so that it will also be satisfied if $\varphi_i^*(x) = 0$. Thus $\psi_i(x, y)$ is independent of x, and that argument will be dropped. This reduces equation (11) to the form

$$K''(x, y) = \sum_{i=1}^{r} \varphi_i^*(x)\psi_i(y), \tag{17}$$

and equations (14) and (15) become equivalent to

$$\int_a^b \varphi_i(x) d\psi_j(y) = \delta_{ij}, \qquad (i, j = 1, 2, \cdots, r), \tag{18}$$

where $\delta_{ij} = 1$ or 0, according as $i = j$ or $i \neq j$.

§ 4. **Application to the Transformation** B_λ.—The theory developed in the preceding section can be applied immediately to the transformation $B_\lambda = E - \lambda A$. If the critical values are represented by $\lambda_1, \lambda_2, \cdots$, equation (17) can be replaced by

$$K''_\alpha(x, y) = \frac{1}{\lambda_\alpha} \sum_{i=1}^{r_\alpha} \varphi_{\alpha i}^*(x)\psi_{\alpha i}(y), \qquad (\alpha = 1, 2, \cdots),$$

where the $\varphi_{\alpha i}^*$ are null elements with respect to $\lambda_\alpha A$ determined as in § 3, and equation (18) by

$$\int_a^b \varphi_{\alpha i}(y) d\psi_{\alpha j}(y) = \delta_{ij}, \qquad (i, j = 1, 2, \cdots, r_\alpha). \tag{19}$$

The transformations $A_{\alpha i}$, corresponding to the A_i in the preceding section, vanish for elements which are kernel with respect to λ_α, and reduce null elements to null elements. It has been proved in § 2 that an element which is null for one value of λ is kernel for every other value. Consequently the equations

$$\int_a^b \varphi_{\alpha i}(y) d\psi_{\beta j}(y) = 0, \quad (i = 1, 2, \cdots, r_\alpha;\ j = 1, 2, \cdots, r_\beta;\ \beta \neq \alpha), \tag{20}$$

must be satisfied. Similarly each of the transformations

$$A''_\alpha(f) = \int_a^b f(y) d_y K''_\alpha(x, y), \qquad (\alpha = 1, 2, \cdots),$$

is orthogonal to each of the others. Since by definition A''_α is orthogonal to $A - A''_\alpha$, it must be orthogonal to $A - \sum_{\beta=1}^{n} A''_\beta$, $(n = \alpha, \alpha + 1, \cdots)$. It now follows from a theorem in § 1 that the transformation $E - \lambda[A - \sum_{\alpha=1}^{n} A''_\alpha]$ is regular for all values of λ excepting $\lambda_{n+1}, \lambda_{n+2}, \cdots$, where it has the same null elements as B_λ.

If the series

$$\sum_{\alpha=1}^{\infty} K''_\alpha(x, y) = \sum_{\alpha=1}^{\infty} \frac{1}{\lambda_\alpha} \sum_{i=1}^{r_\alpha} \varphi^*_{\alpha i}(x) \psi_{\alpha i}(y),$$

converges in such a way that

$$\operatorname*{limit}_{n \to \infty} V_y[\sum_{\alpha=n}^{\infty} K''_\alpha(x, y)] = 0$$

uniformly, the series of transformations $\sum_{\alpha=1}^{\infty} A''_\alpha$ will converge uniformly, and vice versa. Then since $\sum_{\alpha=1}^{n} A''_\alpha$ is orthogonal to $A - \sum_{\alpha=1}^{n} A''_\alpha$ for all values of n, their limits must also be orthogonal, and also $A - \sum_{\alpha=1}^{n} A''_\alpha$ must be orthogonal to each of the transformations A''_α. Consequently all of the null elements with respect to B_λ are null with respect to $E - \lambda \sum_{\alpha=1}^{\infty} A''_\alpha$, and the transformation $E - \lambda[A - \sum_{\alpha=1}^{\infty} A''_\alpha]$ is regular for all values of λ.

§ 5. **The Stieltjes Integral Equation of the First Kind.**—In this section it will be assumed that $g(x)$ is a known function which can be expanded into a uniformly convergent series

$$g(x) = \sum_{\alpha=1}^{\infty} \sum_{i=1}^{r_\alpha} g_{\alpha i} \varphi_{\alpha i}(x), \tag{21}$$

where the $g_{\alpha i}$'s are constants, and conditions will be found under which there is a solution of the equation

$$g(x) = \int_a^b f(y) d_y K(x, y), \tag{22}$$

which can be expanded into a similar convergent series

$$f(x) = \sum_{\alpha=1}^{\infty} \sum_{i=1}^{r_\alpha} f_{\alpha i} \varphi_{\alpha i}(x). \tag{23}$$

The function $K(x, y)$ will be assumed to satisfy the conditions for complete continuity. The constants $g_{\alpha i}$ can be determined from the equations

$$\int_a^b g(x) d\psi_{\alpha i}(x) = \int_a^b \sum_{\beta=1}^{\infty} \sum_{j=1}^{r_\beta} g_{\beta j} \varphi_{\beta j}(x) d\psi_{\alpha i}(x).$$

The series in the right members of the above are uniformly convergent, and consequently can be integrated term by term, and equations (19) and (20) reduce them to

$$\int_a^b g(x) d\psi_{\alpha i}(x) = g_{\alpha i}, \qquad (i = 1, 2, \cdots, r_\alpha; \ \alpha = 1, 2, \cdots).$$

If equations (21) and (23) are substituted in equation (22), and the right member integrated term by term, it becomes

$$\sum_{\alpha=1}^{\infty} \sum_{i=1}^{r_\alpha} g_{\alpha i} \varphi_{\alpha i}(x) = \sum_{\alpha=1}^{\infty} \sum_{i=1}^{r_\alpha} f_{\alpha i} A(\varphi_{\alpha i}) = \sum_{\alpha=1}^{\infty} \frac{1}{\lambda_\alpha} \sum_{i=1}^{r_\alpha} f_{\alpha i} \varphi^*_{\alpha i}(x),$$

with the $\varphi^*_{\alpha i}$ defined as in § 3 by equations such as

$$\varphi^*_{\alpha i} = \varphi_{\alpha i} - \sum_{j=1}^{i-1} a_{\alpha i j} \varphi_{\alpha j}.$$

The constants $f_{\alpha i}$ are then determined by the systems of linear equations

$$f_{\alpha j} - \sum_{i=j+1}^{r_\alpha} a_{\alpha i j} f_{\alpha i} = \lambda_\alpha g_{\alpha j}, \qquad (j = 1, 2, \cdots, r_\alpha; \ \alpha = 1, 2, \cdots).$$

Since the determinant of each system is unity, the constants $f_{\alpha i}$ are determined uniquely, and if the series (23) converges uniformly, it will furnish the required solution of equation (22).

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EQUIVALENCE AND REDUCTION OF PAIRS OF HERMITIAN FORMS.*

BY MAYME IRWIN LOGSDON.

Introduction and Summary.—Of fundamental importance in the theory of matrices and forms has been the use by Weierstrass, Kronecker, and Frobenius of the theory of *elementary divisors* in the study of equivalence and reduction of pairs of bilinear and pairs of quadratic forms.

In this paper a generalization is made in that the basal theorems of the theory are extended to any hermitian λ-matrix, i.e., a matrix whose elements are polynomials of degree n in λ with coefficients in the field of complex numbers and are such that the conjugate of the element in the ith row and jth column is equal to the element in the jth row and ith column for $i, j = 1, 2, \cdots, n$.

Inasmuch as a linear substitution with matrix $\boldsymbol{P}$ on the variables of an hermitian form with matrix $\boldsymbol{a}$ gives a form with matrix $\boldsymbol{b} = \overline{\boldsymbol{P}}'\boldsymbol{aP}$ where $\overline{\boldsymbol{P}}$ means the matrix formed from $\boldsymbol{P}$ by taking the conjugate imaginary of each element and $\boldsymbol{P}'$ means the transposed matrix $\boldsymbol{P}$, the extension of the general theory to the hermitian λ-matrix is justified by the proof in Part I of

THEOREM II. *If $b = \bar{p}'aq$ where p and q are non-singular and independent of λ, and where a and b are hermitian λ-matrices, then there exists a matrix P such that $b = \overline{P}'aP$.*

The special application is made to hermitian λ-matrices whose elements are linear in λ. Such will be the matrix of the pencil of forms

$$\lambda A - B = \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda a_{ij} - b_{ij})\bar{x}_i x_j.$$

The coincidence of the elementary divisors is found to be a necessary and sufficient condition for the equivalence of two pairs of hermitian matrices free of λ and for the equivalence of two pairs of hermitian forms.

In Part II, the Weierstrass reduction is shown to hold in case one of the forms is definite, a condition which insures reality of all the elementary divisors; in fact the Weierstrass method can be used for finding the contribution to the canonical form of any *real* elementary divisor. In the case however of conjugate complex elementary divisors, $(\lambda - \bar{a})^e$ and $(\lambda - a)^e$, it was found necessary and possible to regularize the λ-matrix with respect to the two conjugate imaginary linear factors simultaneously

* Presented to the Society at Chicago, March 25, 1921.

and also to expand the terms representing the adjoint form with respect to these two factors simultaneously.

In the actual work of reduction use was made of an algebraic simplification suggested and described by Dickson* in "Pairs of Bilinear or Quadratic Forms." The importance of this simplification is that in the case of bilinear forms the reduction may be accomplished rationally while in the case of quadratic or hermitian forms the computations are simplified if the constants c_κ which appear are held until the last step of the reduction before being absorbed in the variables.

The canonical form obtained † is given in Part I, Theorem V.

PART I—EQUIVALENCE.

We seek the conditions for equivalence of two pairs of hermitian forms:

$$A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\bar{x}_i x_j, \qquad A^* = \sum_{i=1}^{n} \sum_{j=1}^{n} a^*_{ij}\bar{x}_i x_j,$$

$$B = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\bar{x}_i x_j, \qquad B^* = \sum_{i=1}^{n} \sum_{j=1}^{n} b^*_{ij}\bar{x}_i x_j,$$

with the respective matrices $\boldsymbol{a}$, $\boldsymbol{a}^*$, $\boldsymbol{b}$, $\boldsymbol{b}^*$ of which $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are non-singular, and where $\bar{a}_{ij} = a_{ji}$, etc., for $i, j = 1, \cdots, n$.

If a linear transformation with non-singular matrix, $\boldsymbol{c}$, whose elements are independent of λ,

$$x_i = \sum_{j=1}^{n} c_{ij} y_j \qquad (i = 1, \cdots, n),$$

with the induced transformation on the conjugate variables,

$$\bar{x}_i = \sum_{j=1}^{n} \bar{c}_{ij} \bar{y}_j \qquad (i = 1, \cdots, n),$$

be applied to A and B, the transformed hermitian forms will have the respective matrices $\bar{\boldsymbol{c}}'\boldsymbol{ac}$ and $\bar{\boldsymbol{c}}'\boldsymbol{bc}$. If, now, these are to be the forms A^* and B^*, we must have

$$\boldsymbol{a}^* = \bar{\boldsymbol{c}}'\boldsymbol{ac}, \qquad \boldsymbol{b}^* = \bar{\boldsymbol{c}}'\boldsymbol{bc}. \tag{1}$$

We shall show that a necessary and sufficient condition that equations (1) be satisfied is that the two hermitian λ-matrices, $\boldsymbol{m} = \boldsymbol{a} - \lambda\boldsymbol{b}$ and $\boldsymbol{m}^* = \boldsymbol{a}^* - \lambda\boldsymbol{b}^*$, have the same elementary divisors. We state the problem thus:

* Transactions A. M. Society, v. 10, 1909, p. 350.

† In the Proceedings of the London Mathematical Society, v. 32, 1900, pp. 321——, Bromwich obtains such a reduction by a special device, stating that "apparently this method (the Frobenius-Weierstrass method) cannot be extended so as to cover the analogous theory for conjugate imaginary substitutions, which would be applied to a pair of Hermite's forms."

Given any two hermitian λ-matrices, m and m^*, with elements polynomials in λ, to find necessary and sufficient conditions for the existence of a non-singular matrix c with elements independent of λ, such that $m^* = \bar{c}'mc$.

In proof we must show that if two hermitian λ-matrices are equivalent, the corresponding hermitian forms may be obtained, the one from the other, by a non-singular transformation on the variables. The first step of the proof will consist in establishing

THEOREM I. *If two hermitian λ-matrices, $m = a - \lambda b$ and $m^* = a^* - \lambda b^*$, are equivalent,* there exist two non-singular matrices, t and q, whose elements are independent of λ, such that*

$$m^* = tmq. \tag{2}$$

Proof: By the equivalence of m and m^* there exist non-singular λ-matrices t_0 and q_0 with determinants free of λ, such that

$$m^* = t_0 m q_0. \tag{3}$$

Now divide t_0 by m^* and $(q_0)^{-1}$ by m † in such a way as to get matrices t_1, t, s_1, s which satisfy the relations

$$t_0 = m^* t_1 + t, \qquad (q_0)^{-1} = s_1 m + s, \tag{4}$$

t and s being matrices whose elements do not involve λ. From (3) we get $t_0 m = m^* q_0^{-1}$. Substituting here from (4) we have

$$m^*(t_1 - s_1)m = m^* s - tm. \tag{5}$$

Now the right member of (5) is a λ-matrix of at most the first degree, while for $t_1 - s_1 \neq 0$ the left member would be of at least the second degree. Hence $t_1 = s_1$ and

$$m^* s = tm. \tag{6}$$

Whence if we knew that s (and likewise t from (6)) were non-singular, we could write

$$m^* = tms^{-1} \tag{7}$$

and the theorem would be proved.

We proceed to show that s is non-singular. Substitute in the identity $I = q_0 q_0^{-1}$ for q_0^{-1} from (4_2) and get

$$I = q_0 s_1 m + q_0 s. \tag{8}$$

Now divide q_0 by m^* in such a way as to get

$$q_0 = q_1 m^* + q, \tag{9}$$

* Two λ-matrices, m and m^*, are called equivalent (Bôcher, Introduction to Higher Algebra, p. 274) if there exist λ-matrices t_0 and q_0 each having as determinant a number not zero independent of λ such that $m^* = t_0 m q_0$.

† Bôcher, p. 278.

where q is a matrix with elements free of λ. Substituting this value in (8) we have

$$I = q_0 s_1 m + q_1 m^* s + qs$$

which, by use of (6), may be written

$$I - qs = (q_0 s_1 + q_1 t) m. \tag{10}$$

Since the left member does not contain λ we must have $q_0 s_1 + q_1 t$ identically zero, and therefore

$$I = qs. \tag{11}$$

s is then non-singular, and we may write (6) in the form *

$$m^* = tmq.$$

Since $p = \bar{t}'$ is evidently a λ-matrix whose determinant equals the conjugate of the determinant t, we may express the definition of equivalence in the following form which is more convenient for hermitian forms:

Two hermitian λ-matrices, m and m^*, are *equivalent* if there exist non-singular matrices p and q with determinants free of λ, such that $m^* = \bar{p}'mq$.

The second step in solution of the original problem makes possible the extension of the theory of equivalence to the corresponding forms. It consists in proving

THEOREM II. *If $b = \bar{p}'aq$, where p and q are non-singular and independent of λ, and where a and b are hermitian λ-matrices, then there exists a matrix P such that*

$$b = \bar{P}'aP$$

and such that P depends not on a or b but solely on p and q.

We have by hypothesis

$$b = \bar{p}'aq, \tag{1}$$

whence, since a and b are hermitian,

$$b = \bar{q}'ap.$$

Equating these two values of b we get

$$\bar{q}'ap = \bar{p}'aq, \tag{2}$$

from which

$$(\bar{q}')^{-1}\bar{p}'a = apq^{-1}. \tag{3}$$

If now we set $U = (\bar{q}')^{-1}\bar{p}'$, then $\bar{U}'$ will be pq^{-1} and (3) becomes

$$Ua = a\bar{U}. \tag{4}$$

* This theorem holds if m and m^* have elements of degree p in λ.

From this we get at once

$$U^2a = a\overline{U}'^2; \tag{5}$$

and, in general,

$$U^ka = a\overline{U}'^{(k)}. \tag{6}$$

From $a = a$ and (4), (5), (6) by using any set of arithmetical multipliers we get

$$\chi(U)a = a\chi(\overline{U}'), \tag{7}$$

where $\chi(U)$ is any polynomial in U. Thus

$$a = [\chi(U)]^{-1}a\chi(\overline{U}'). \tag{7'}$$

Now we choose the polynomial $\chi(U) = V$ so that $V^2 = U$ and so that V is non-singular.* We have then from (7′)

$$a = V^{-1}a\overline{V}'. \tag{8}$$

Substituting (8) in (1) we get

$$b = p'V^{-1}a\overline{V}'q. \tag{9}$$

Now set $P = \overline{V}'q$, then $\overline{P}' = \overline{q}'V$ and $b = \overline{P}'aP$, as desired. For, from the definitions of U and V we have

$$U = V^2 = (\overline{q}')^{-1}\overline{p}'$$

from which we easily obtain $\overline{q}'V = \overline{p}'V^{-1}$.

These theorems permit the use of the general theory of λ-matrices. We state the theorems and definitions which are needed in the sequel:

I. If a and b are equivalent hermitian λ-matrices of rank r, and if $D_i(\lambda)$ is the greatest common divisor of the i-rowed determinants ($i \leqq r$) of a, then it is also the greatest common divisor of the i-rowed determinants of b.

II. Every hermitian λ-matrix of order n and rank r can be reduced by elementary transformations † to the normal form

$$\left\|\begin{array}{cccccc} E_1(\lambda) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & E_2(\lambda) & \cdots & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & E_r(\lambda) & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{array}\right\|,$$

where the coefficient of the highest power of λ in each of the polynomials $E_i(\lambda)$ is unity, and $E_i(\lambda)$ is a factor of $E_{i+1}(\lambda)$ for $i = 1, 2, \cdots, r-1$.

III. The greatest common divisor of the i-rowed determinants of an

* V is in fact a polynomial in U of degree less than n. See Bôcher, l.c., p. 299.

† Elementary transformations as defined in Bôcher, l.c., p. 262.

hermitian λ-matrix of rank r, when $i \leqq r$, is

$$D_i(\lambda) = E_1(\lambda) E_2(\lambda) \cdots E_i(\lambda),$$

where the E's are the polynomials of the last theorem.

IV. A necessary and sufficient condition that two hermitian λ-matrices of order n be equivalent is that they have the same rank r, and that for every value of i from 1 to r inclusive, the i-rowed determinants of one matrix have the same greatest common divisor as the i-rowed determinants of the other.

From the definition of the D's in III, we see that

$$E_i(\lambda) = \frac{D_i(\lambda)}{D_{i-1}(\lambda)}$$

$$(i = 1, 2, \cdots, r), \qquad (D_0(\lambda)) = 1).$$

Hence since the D's with the rank form a complete system of invariants, since the D's completely determine the E's as well as the elementary divisors, and since conversely the D's are completely determined by the E's or by the elementary divisors, we may state thus the

FUNDAMENTAL THEOREM: *A necessary and sufficient condition that two hermitian λ-matrices be equivalent is that they have the same rank and that the elementary divisors of one be identical respectively with the elementary divisors of the other.*

Definition: Two pairs of hermitian matrices a, b and a^*, b^* with elements free of λ will be called *equivalent* if there exist two non-singular matrices p and q, also with elements not involving λ, such that

$$(1) \qquad a^* = \bar{p}' a q, \qquad b^* = \bar{p}' b q.$$

From this definition, Theorem I and the fundamental theorem we have

THEOREM III. *If a, b and a^*, b^* are two pairs of hermitian matrices independent of λ, and if b and b^* are non-singular, a necessary and sufficient condition that these two pairs of matrices be equivalent is that the two λ-matrices*

$$m = a - \lambda b, \qquad m^* = a^* - \lambda b^*,$$

have the same elementary divisors.

Referring now to Theorem II, equivalence conditions for two pairs of matrices may be stated as follows:

THEOREM IV. *If a, b, a^*, b^* are hermitian matrices independent of λ, and if b and b^* are non-singular, a necessary and sufficient condition that a non-singular matrix P exist such that*

$$a^* = \bar{P}' a P, \qquad b^* = \bar{P}' b P,$$

is that the matrices $a - \lambda b$ and $a^* - \lambda b^*$ have the same elementary divisors.

If in particular $b^* = b = I$, where I is the unit matrix, we have

$$I = \bar{P}'P$$

which defines an *orthogonal* hermitian matrix.

Corollary: If the characteristic matrices of a and a^* have the same elementary divisors there will exist an orthogonal matrix P such that

$$a^* = \bar{P}'aP \qquad \text{or} \qquad a^* = P^{-1}aP,$$

i.e., a^* is the transform of a by the orthogonal matrix P.

We have thus obtained the desired conditions for equivalence of two pairs of hermitian forms as stated in the first paragraph of Part I.

If the matrix of the form B is the unit matrix, referring to the last corollary we see that a transformation on the variables with orthogonal matrix P will transform the form B into B^* also with unit matrix. The λ-matrices

$$a - \lambda I, \qquad a^* - \lambda I$$

are now the characteristic matrices of the forms A and A^*, and as before a necessary and sufficient condition for the equivalence of the forms under orthogonal transformation is the coincidence of the elementary divisors of the characteristic matrices of the forms.

If B is a non-singular definite form, a preliminary transformation will transform it to the sum of hermitian squares, $\sum_1^n \bar{x}_i x_i$ with unit matrix, and since the roots of the determinant $|a - \lambda b| = 0$ are now the roots of the characteristic equation of a which are known to be always real,† we have the

Corollary: The elementary divisors of the pencil of hermitian forms $A - \lambda B$, where B is non-singular definite, are all real and of the first degree.‡

As in the quadratic case we have a further

*Corollary:** If A and B are hermitian forms and B is non-singular, a necessary and sufficient condition that it be possible to reduce A and B simultaneously by a non-singular transformation to forms into which only square terms (hermitian squares) enter is that all the elementary divisors of the pair of forms be of the first degree.

Finally, if both matrices are singular but at least one matrix of the pencil $\lambda_1 a + \lambda_2 b$ is non-singular, we may proceed as in the quadratic case § and obtain the desired canonical form.

The canonical form.

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* See Bôcher, l.c., p. 305, for the corresponding theorem for quadratic forms.

† G. Kowalewski-Einfuhrung in die Determinanten Theorie, p. 130.

‡ The proof in Bôcher, l.c., p. 170, for quadratic forms is applicable here.

§ Muth, l.c., p. 87.

THEOREM V. *If $c_1, c_2, \cdots, c_f$ are any real constants including zero, equal or unequal, if $a_g, a_h, \cdots, a_r$ are any complex numbers, equal or unequal, and if $e_1, e_2, \cdots, e_r$ are positive integers such that $e_1 + e_2 + \cdots + e_f + 2e_g + \cdots + 2e_r = n$, there exist pairs of hermitian forms in n variables, the first form being non-singular, which have the elementary divisors*

$$(\lambda - c_1)^{e_1},\ (\lambda - c_2)^{e_2},\ \cdots,\ (\lambda - c_f)^{e_f},\ (\lambda - \bar{a}_g)^{e_g},\ (\lambda - a_g)^{e_g},\ \cdots,\ (\lambda - \bar{a}_r)^{e_r},\ (\lambda - a_r)^{e_r}.$$

In proof after setting $e_1 + e_2 + \cdots + e_f = e$ we exhibit the forms

$$A = \sum_{i=1}^{e_1} \overline{X}_i X_{e_1-i+1} + \sum_{i=e_1+1}^{e_1+e_2} \overline{X}_i X_{2e_1+e_2-i+1} + \cdots + \sum_{i=e-e_f+1}^{e} \overline{X}_i X_{2e-i+1} + \sum_{j=e+1}^{e+2e_g} \overline{X}_j X_{2e+2e_g-j+1} + \cdots + \sum_{j=n-2e_r+1}^{n} \overline{X}_j X_{2n-2e_r-j+1}.$$

$$\begin{aligned} B = {} & \sum_{i=1}^{e_1} c_1 \overline{X}_i X_{e_1-i+1} + \sum_{i=e_1+1}^{e_1+e_2} c_2 \overline{X}_i X_{2e_1+e_2-i+1} + \cdots \\ & + \sum_{i=e-e_f+1}^{e} c_f \overline{X}_i X_{2e-i+1} + \sum_{i=1}^{e_1-1} \overline{X}_i X_{e_1-i} + \sum_{i=e_1+1}^{e_1+e_2-1} \overline{X}_i X_{2e_1+e_2-i} + \cdots \\ & + \sum_{j=e+1}^{e+e_g} a_g \overline{X}_j X_{2e+2e_g-j+1} + \sum_{j=e+e_g+1}^{e+2e_g} \bar{a}_g \overline{X}_j X_{2e+2e_g-j+1} + \cdots \\ & + \sum_{j=n-2e_r+1}^{n-e_r} a_r \overline{X}_j X_{2n-2e_r-j+1} + \sum_{j=n-e_r+1}^{n} \bar{a}_r \overline{X}_j X_{2n-2e_r-j+1} \\ & + \sum_{j=e+1}^{e+2e_g-1} \overline{X}_j X_{2e+2e_g-j} + \cdots + \sum_{j=n-2e_r+1}^{n-1} \overline{X}_j X_{2n-2e_r-j}. \end{aligned}$$

PART II—REDUCTION.

In the attempt to apply the methods of Weierstrass to reducing a pair of hermitian forms:

$$A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \bar{x}_i x_j \qquad \text{and} \qquad B = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \bar{x}_i x_j,$$

where $\bar{a}_{ij} = a_{ji}$, $\bar{b}_{ij} = b_{ji}$, $|a_{ij}| \neq 0$, no trouble arises in finding the contribution to the canonical form due to any real linear factor of the λ-matrix, $\lambda \boldsymbol{a} - \boldsymbol{b}$, though proof is needed that certain theorems are actually extensible to this type of matrix. In dealing with complex linear factors however an essential modification is required. We shall first indicate the main steps in the process of reduction, then study in detail the separate cases where difference of treatment is required. Wherever the Weierstrass treatment as given in Muth's "Theorie und Anwendung der Elementartheiler" is valid without separation of the two cases, the details of the work will be omitted

and reference to Muth given. The following notations and definitions will be used:

S = determinant of the form $C = \lambda A - B$.

l_κ = exponent of the linear factor $(\lambda - r)$ in $D_\kappa(\lambda)$, i.e., in the greatest common divisor of all the κ-rowed minor determinants of S. There will be at least one κ-rowed minor determinant of S which contains $(\lambda - r)$ exactly l_κ times and is then defined to be *regular* with respect to this factor. We have

$$l_1 \leqq l_2 \leqq \cdots \leqq l_n$$

and also

$$l_i - l_{i-1} = e_i,$$

where $(\lambda - r)^{e_i}$ is an elementary divisor.

S_{ij} = cofactor of the element $\lambda a_{ij} - b_{ij}$ in S.

$S^{(\kappa)}$ = principal minor determinant with $n - \kappa$ rows obtained from S by deleting the first κ rows and the first κ columns. We note $S^{(\kappa)} = S_{\kappa\kappa}{}^{(\kappa-1)}$. For $\kappa = 0$, we define $S^{(0)} = S$, and for $\kappa = n$, $S^{(n)} = 1$.

S_ρ = the principal ρ-rowed minor determinant in the upper left-hand corner of the matrix.

$S_{\rho;\, ik}$ = $(\rho + 1)$-rowed minor determinant obtained by bordering S_ρ by the ith row and the kth column of the original matrix.

The main steps in the reduction are:

(1) Any hermitian λ-matrix may by elementary transformations* be regularized with respect to

(1) any real linear factor, $(\lambda - c)$.

(2) any pair of conjugate imaginary linear factors,

$$(\lambda - \bar{a}), \quad (\lambda - a).$$

* Elementary transformations of an hermitian λ-matrix are defined as follows:

1st. Interchanging two columns and the same two rows.

2d. Multiplying the elements of a column by a number m independent of λ and then multiplying the elements of the corresponding row by the conjugate of m.

3d. Adding to the elements of the kth column the products of the corresponding elements of the jth column $(j \neq k)$ by a polynomial, $\psi(\lambda)$, then adding to the elements of the kth row the products of the corresponding elements of the jth row each multiplied by the conjugate, $\overline{\psi}(\lambda)$, of the polynomial $\psi(\lambda)$.

The elementary transformations of the variables of the corresponding hermitian form which effect on the matrix of the form the above defined transformations are:

1st. $x_i = y_i \quad (i = 1, \cdots, n;\ i \neq j,\ i \neq k)$
$x_j = y_k$
$x_k = y_j$.

2d. $x_i = y_i \quad (i = 1, \cdots, n;\ i \neq j)$
$x_j = m y_j$.

3d. $x_i = y_i \quad (i = 1, \cdots, n;\ i \neq j)$
$x_j = y_j + \psi(\lambda) y_k$.

Definition: A matrix S is *regular* with respect to a real linear factor or with respect to a pair of conjugate imaginary linear factors if each of the principal minor determinants obtained from S by deleting the first k rows and columns, $(k = 1, 2, \cdots, n-1)$, is so.

(2) The adjoint form may be written as a sum of terms in which every factor in a denominator is regular with respect to a given linear factor (or pair of factors); viz.,

$$\sum_{i,j}^{1,n} \frac{S_{ij}}{S} \bar{u}_j u_i = \frac{\bar{X}'X'}{SS'} + \frac{\bar{X}''X''}{S'S''} + \cdots + \frac{\bar{X}^{(n)}X^{(n)}}{S^{(n-1)}S^{(n)}}. \tag{1}$$

(3) 1st. Each term on the right of (1) may be expanded with respect to a real linear factor, $(\lambda - c)$, the total coefficients of $1/(\lambda - c)^2$, $1/(\lambda - c)$ secured; finally, the total coefficients of $1/\lambda^2$, $1/\lambda$ secured. This will give the contribution of this particular linear factor, $\lambda - c$, to the canonical form.

2d. Each term on the right of (1) may be expanded with respect to $\lambda - \bar{a}$, $\lambda - a$ simultaneously and the total coefficients of $1/\lambda^2$, $1/\lambda$ secured.

(4) The adjoint form may be expanded by determinantal methods in descending powers of λ and the coefficients of $1/\lambda^2$, $1/\lambda$ obtained. These prove to be $\boldsymbol{B}$ and $\boldsymbol{A}$ respectively.

(5) A comparison of the results of (3) and (4) with Theorem V of Part I gives the desired expression for $\boldsymbol{A}$ and $\boldsymbol{B}$.

Proofs:

(I_1). Any hermitian λ-matrix, $\boldsymbol{S}$, may by elementary transformations be regularized with respect to a real linear factor, $\lambda - c$.

In proof we must show

(*a*) Every regular ρ-rowed minor determinant $(\rho > 1)$ contains at least one regular $(\rho - 1)$-rowed minor determinant as first minor.

(*b*) Every regular $(\rho - 1)$-rowed minor determinant $(\rho > 2)$ is contained in at least one regular ρ-rowed minor determinant as first minor.

(*c*) If a $(\rho - 1)$-rowed principal minor, $S_{\rho-1}$, is regular, but no ρ-rowed principal minor, $S_{\rho-1;\,i,\,i}$, containing it is regular, there is an elementary transformation of the variables which without disturbing the regularity of any S_k $(k = 1, \cdots, \rho - 1)$ will so transform the matrix $\boldsymbol{S}$ that there will be a regular ρ-rowed principal minor containing $\boldsymbol{S}_{\rho-1}$ as a first minor.

For proof of (*a*) and (*b*) see Muth, l.c., pp. 7–11.

Proof of (*c*): The existence of a regular ρ-rowed minor, $\boldsymbol{S}_{\rho-1;\,j,\,k}$, containing $\boldsymbol{S}_{\rho-1}$, is guaranteed by (*b*) above. Now apply to the variables of the form a preliminary transformation which will interchange the ρth and jth rows and columns and the $(\rho + 1)$st and kth rows and columns. We now have $\boldsymbol{S}_{\rho-1;\,\rho,\,\rho+1}$ for our regular ρ-rowed minor. Now apply the trans-

formation

$$T_m\colon x_i = y_i \qquad (i = 1, \cdots, n;\ i \neq \rho + 1),$$
$$x_{\rho+1} = y_{\rho+1} - m y_\rho.$$

The effect on the matrix is to subtract the products by m of the elements of the $(\rho + 1)$st column from the elements of the ρth column and then the products by $\bar{m}$ of the elements of the $(\rho + 1)$st row from the elements of the ρth row. Obviously S_k is not changed $(k = 1, \cdots, \rho - 1)$, but the principal minor, call it S'_ρ, of order ρ and containing $S_{\rho-1}$, is now regular. For we have

$$S'_\rho = S_\rho - mS_{\rho-1;\,\rho,\,\rho+1} - \bar{m}S_{\rho-1;\,\rho+1,\,\rho} + \bar{m}mS_{\rho-1;\,\rho+1,\,\rho+1}. \tag{2}$$

Now S_ρ and $S_{\rho-1;\,\rho+1,\,\rho+1}$ each by hypothesis contains $\lambda - c$ more than l_ρ times; $S_{\rho-1;\,\rho,\,\rho+1}$ contains $\lambda - c$ exactly l_ρ times; $S_{\rho-1;\,\rho+1,\,\rho}$ is the conjugate of $S_{\rho-1;\,\rho,\,\rho+1}$ and therefore contains the real linear factor $\lambda - c$ exactly l_ρ times. It remains then to show that the sum $R = mS_{\rho-1;\,\rho,\,\rho+1} + \bar{m}\bar{S}_{\rho-1;\,\rho,\,\rho+1}$ does not have a higher power of $\lambda - c$ as factor than each part, for every m. Divide R by $(\lambda - c)^{l_\rho}$. $R = (\lambda - c)^{l_\rho}[mf(\lambda) + \bar{m}\bar{f}(\lambda)]$, and, since $f(c)$ is not zero, if we set $m = 1/f(c)$ we have $mf(c) + \bar{m}\bar{f}(c)$ not zero. Thus S'_ρ contains $\lambda - c$ exactly l_ρ times and is regular, as stated.

(I_2) Any hermitian λ-matrix, S, may by elementary transformations be regularized with respect to an imaginary linear factor. When this is done the matrix will be also regular with respect to the conjugate imaginary linear factor.

As before, (a) and (b) may be assumed for the factor $\lambda - a$ from the proof for bilinear forms. To prove (c) we apply the same transformation, T_m, and get as before the right member of equation (3). Remembering that the first and last terms are principal minors and hence have real coefficients, and that neither is regular with respect to $(\lambda - \bar{a})(\lambda - a)$, we may factor the right member of (3) thus:

$$[(\lambda - \bar{a})(\lambda - a)]^{l_\rho}[(\lambda - \bar{a})(\lambda - a)]^r f_1(\lambda) - m(\lambda - \bar{a})^p f_2(\lambda)$$
$$- \bar{m}(\lambda - a)^p \bar{f}_2(\lambda) + \bar{m}m[(\lambda - \bar{a})(\lambda - a)]^s f_3(\lambda),$$

where $r > 0$, $s > 0$, $p \geqq 0$, $f_2(a) \neq 0$.

If $p \neq 0$, the theorem is proved since $\lambda - a$ is a factor of all the terms in the brackets but one and consequently is not a factor of the sum. Thus S'_ρ is regular, as stated. If $p = 0$, we must show that $mf_2(\lambda) + \bar{m}\bar{f}_2(\lambda)$ is not divisible by $\lambda - a$ where $f_2(a) \neq 0$ and $\bar{f}_2(\bar{a}) \neq 0$, i.e., we must show that m can be so chosen that $mf_2(a) + \bar{m}\bar{f}(a) \neq 0$. This is the same condition reached before and is satisfied by $m = 1/f_2(a)$. Hence we may use the Jacobi transformation of the adjoint form, viz.,*

$$\sum_{i,j}^{1,n} \frac{S_{ij}}{S}\,\bar{u}_j u_i = \frac{\bar{X}'X'}{SS'} + \frac{\bar{X}''X''}{S'S''} + \cdots + \frac{\bar{X}^{(n)}X^{(n)}}{S^{(n-1)}S^{(n)}}, \tag{3}$$

* Muth, l. c., pp. 70–72.

with a determinant regular with respect to any linear factor, and where

$$
\begin{aligned}
X' &= S_{11}u_1 + S_{21}u_2 + S_{31}u_3 + \cdots + S_{n1}u_n \\
X'' &= \phantom{S_{11}u_1 + {}} S'_{22}u_2 + S'_{32}u_3 + \cdots + S'_{n2}u_n \\
X''' &= \phantom{S_{11}u_1 + S_{21}u_2 + {}} S''_{33}u_3 + \cdots + S''_{n3}u_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
X^{(n)} &= \phantom{S_{11}u_1 + S_{21}u_2 + S_{31}u_3 + \cdots + {}} S^{(n-1)}_{nn}u_n
\end{aligned} \tag{4}
$$

(3) To decompose the general term on the right of (3) with respect to the real linear factor, $\lambda - c$, we may write

$$\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{S^{(\kappa-1)}S^{(\kappa)}} = \frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{(\lambda - c)^{l_{\kappa-1}+l_\kappa}} \cdot \frac{1}{C_\kappa q^2}, \text{*} \tag{5}$$

where C_κ is the coefficient of the highest power of λ in $S^{(\kappa-1)}S^{(\kappa)}$.† It is evident that q^2, a polynomial in λ with real coefficients, will not contain $\lambda - c$ as a factor since $S^{(\kappa-1)}$ and $S^{(\kappa)}$ are regular, and we may then expand $X^{(\kappa)}/q$ and $\overline{X}^{(\kappa)}/q$ in power series in $\lambda - c$. Also, since in the definition of $X^{(\kappa)}$ in (4) the coefficient $S^{(n-\kappa)}_{\kappa\kappa}$ of u_κ is regular with respect to $\lambda - c$, $X^{(\kappa)}$, and consequently $\overline{X}^{(\kappa)}$ will contain $\lambda - c$ exactly l_κ times. Thus

$$\frac{\overline{X}^{(\kappa)}}{q} = (\lambda - c)^{l_\kappa}[\overline{X}_{\kappa 1} + (\lambda - c)\overline{X}_{\kappa 2} + (\lambda - c)^2\overline{X}_{\kappa 3} + \cdots],$$

$$\frac{X^{(\kappa)}}{q} = (\lambda - c)^{l_\kappa}[X_{\kappa 1} + (\lambda - c)X_{\kappa 2} + (\lambda - c)^2 X_{\kappa 3} + \cdots].$$

Hence

$$\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{S^{(\kappa-1)}S^{(\kappa)}} = \frac{1}{C_\kappa}\frac{1}{(\lambda - c)^{e_\kappa}}[\overline{X}_{\kappa 1} + (\lambda - c)\overline{X}_{\kappa 2} + \cdots][X_{\kappa 1} + (\lambda - c)X_{\kappa 2} + \cdots],$$

where the $X_{\kappa\mu}$ are linearly independent polynomials in c, u_κ, $\cdots$, u_n and the coefficients of the two forms A and B and are homogeneous in the u's. Thus dropping the subscript, κ, we may write the right member

$$\frac{1}{C}\frac{1}{(\lambda - c)^e}[Z_1 + Z_2(\lambda - c) + Z_3(\lambda - c)^2 + \cdots]$$

$$= \cdots + \frac{Z_{e-1}}{C(\lambda - c)^2} + \frac{Z_e}{C(\lambda - c)} + \cdots.$$

Now, defining $F_\kappa = Z_e = \sum_{i=1}^{i=e}\overline{X}_i X_{e-i+1}$ and $G_\kappa = Z_{e-i} = \sum_{i=1}^{i=e-1}\overline{X}_i X_{e-i}$ ($G_\kappa = 0$ for $e_\kappa = 1$) we have

$$\frac{F_\kappa}{C_\kappa(\lambda - c)} + \frac{G_\kappa}{C_\kappa(\lambda - c)^2} = \frac{1}{C_\kappa}\left[\frac{F_\kappa}{\lambda} + \frac{cF_\kappa + G_\kappa}{\lambda^2} + \cdots\right].$$

---

* For convenience in notation l_k shall henceforth represent the exponent of the factor $\lambda - c$ in the greatest common divisor of the $(n - \kappa)$-rowed minor determinants of S. We have then the inequalities, $l_0 \geqq l_1 \geqq l_2 \geqq \cdots \geqq l_n$, with $l_{\kappa-1} - l_\kappa = e_\kappa$.

† See Dickson, l.c.

There will be a similar expression obtained from each of the terms on the right of (3). The total contribution due to the real factor $\lambda - c$ to the canonical form is then obtained by taking the sum

$$\sum_{\kappa=1}^{f}\left[\frac{1}{C_\kappa}\frac{F_\kappa}{\lambda}+\frac{1}{C_\kappa}\frac{cF_\kappa+G_\kappa}{\lambda^2}\right],$$

where f is the number of distinct real linear factors. The numbers $1/C_\kappa$ may be now absorbed in the variables by $x' = x/\sqrt{C_\kappa}$ since the original forms and the transformations used have allowed irrationalities as well as imaginaries.

If now the above process of regularization and expansion be repeated for the remaining real linear factors, the total contribution of these factors is obtained.

To decompose with respect to $(\lambda - \bar{a})(\lambda - a)$ the general term of (3), we need to get the partial fractions of this term which have linear or quadratic denominators, viz., $(\lambda - \bar{a})$, $(\lambda - a)$, $(\lambda - \bar{a})^2$, $(\lambda - a)^2$, $(\lambda - \bar{a})(\lambda - a)$, since these and only these terms in the decomposition will contribute to the coefficient of $1/\lambda$ and $1/\lambda^2$. We set

$$\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{S^{(\kappa-1)}S^{(\kappa)}}=\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{C_\kappa[(\lambda-\bar{a})(\lambda-a)]^{l_{\kappa-1}+l_\kappa}q^2},$$

where q^2 is a polynomial in λ containing neither $\lambda - \bar{a}$ nor $\lambda - a$ as a factor and with unity for the coefficient of the highest power of λ. Now expand $X^{(\kappa)}/q$ in powers of $\lambda - a$ and $\overline{X}^{(\kappa)}/q$ in powers of $\lambda - \bar{a}$. Thus

$$\frac{X^{(\kappa)}}{q}=[(\lambda-\bar{a})(\lambda-a)]^{l_\kappa}[X_{\kappa1}+(\lambda-a)X_{\kappa2}+(\lambda-a)^2X_{\kappa3}+\cdots],$$

$$\frac{\overline{X}^{(\kappa)}}{q}=[(\lambda-\bar{a})(\lambda-a)]^{l_\kappa}[\overline{X}_{\kappa1}+(\lambda-\bar{a})\overline{X}_{\kappa2}+(\lambda-\bar{a})^2\overline{X}_{\kappa3}+\cdots],$$

where the $X_{\kappa\mu}$ are polynomials in $\bar{a}$, a, the coefficients of A and B, and homogeneous in $u_\kappa, \cdots, u_n$, while the $\overline{X}_{\kappa\mu}$ are the corresponding conjugate functions of $\bar{u}_\kappa, \cdots, \bar{u}_n$. We have then

$$\begin{aligned}\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{S^{(\kappa-1)}S^{(\kappa)}}=\frac{1}{C_\kappa}\frac{1}{(\lambda-\bar{a})^{e_\kappa}(\lambda-a)^{e_\kappa}}&[\overline{X}_{\kappa1}+(\lambda-\bar{a})\overline{X}_{\kappa2}+(\lambda-\bar{a})^2\overline{X}_{\kappa3}\\&+\cdots][X_{\kappa1}+(\lambda-a)X_{\kappa2}+(\lambda-a)^2X_{\kappa3}+\cdots].\end{aligned}$$

Omitting all κ subscripts, we may write the right member

$$\frac{1}{C}\left[\frac{\overline{X}_1}{(\lambda-\bar{a})^e}+\frac{\overline{X}_2}{(\lambda-\bar{a})^{e-1}}+\cdots+\frac{\overline{X}_e}{\lambda-\bar{a}}+\overline{X}_{e+1}+\overline{X}_{e+2}(\lambda-\bar{a})+\cdots\right.$$

$$+ \bar{X}_{2e}(\lambda - \bar{a})^{e-1} + \cdots \Big]\Big[\frac{X_1}{(\lambda - a)^e} + \cdots + \frac{X_e}{\lambda - a} + X_{e+1} + \cdots$$
$$+ X_{2e}(\lambda - a)^{e-1} + \cdots \Big]$$
$$= \frac{1}{C}\Big[\frac{\bar{X}_1 X_{2e}(\lambda - a)^{e-1}}{(\lambda - \bar{a})^e} + \frac{\bar{X}_1 X_{2e-1}(\lambda - a)^{e-2}}{(\lambda - \bar{a})^e} + \frac{\bar{X}_2 X_{2e-1}(\lambda - a)^{e-2}}{(\lambda - \bar{a})^{e-1}}$$
$$+ \frac{\bar{X}_2 X_{2e-2}(\lambda - a)^{e-3}}{(\lambda - \bar{a})^{e-1}} + \cdots + \frac{\bar{X}_e X_{e+1}}{\lambda - \bar{a}} + \frac{\bar{X}_e X_e}{(\lambda - \bar{a})(\lambda - a)}$$
$$+ \frac{\bar{X}_{e+1} X_e}{\lambda - a} + \cdots + \frac{\bar{X}_{2e} X_1 (\lambda - \bar{a})^{e-1}}{(\lambda - a)^e} + \cdots \Big]$$
$$= \frac{1}{C}\frac{1}{\lambda}[\bar{X}_1 X_{2e} + \bar{X}_2 X_{2e-1} + \cdots + \bar{X}_e X_{e+1} + \cdots + \bar{X}_{2e} X_1]$$
$$+ \frac{1}{C}\frac{1}{\lambda^2}[a(\bar{X}_1 X_{2e} + \cdots + \bar{X}_e X_{e+1}) + \bar{a}(\bar{X}_{e+1} X_e + \cdots + \bar{X}_{2e} X_1)$$
$$+ \bar{X}_1 X_{2e-1} + \bar{X}_2 X_{2e-2} + \cdots + \bar{X}_{2e-1} X_1].$$

Thus we have found the part of the coefficient of $1/\lambda$ and $1/\lambda^2$ due to one term in the right member of (3) and contributed by the pair of linear factors $(\lambda - \bar{a})(\lambda - a)$ where the e in the last expression is the e_κ which belongs to the elementary divisors $(\lambda - \bar{a})^{e_\kappa}$, $(\lambda - a)^{e_\kappa}$. The total coefficient of $1/\lambda$ is seen to be (after the constant C_κ is absorbed in the variables)

$$\sum_{j=1}^{2e_k} \bar{X}_j X_{2e_k - j + 1}$$

and the total coefficient of $1/\lambda^2$ is

$$\sum_{j=1}^{e_k} a\bar{X}_j X_{2e_k - j + 1} + \sum_{j=e_k+1}^{2e_k} \bar{a}\bar{X}_j X_{2e_k - j + 1}.$$

For other pairs of complex elementary divisors we proceed as here, then adding the coefficients of $1/\lambda$ obtained from all the linear factors, real and imaginary, and adding the coefficients of $1/\lambda^2$ obtained from all the linear factors, real and imaginary, we compare with the coefficients of these same powers of λ obtained by expanding the adjoint form

$$\sum_{i,j}^{1\cdots n} \frac{S_{ij}\bar{u}_j u_i}{S}$$

by determinanta methods,* thus obtaining the desired canonical forms given in Part I, Theorem V.

THE UNIVERSITY OF CHICAGO,
May, 1921.

* Muth, l.c., p. 81. $\sum_{i,j}^{1\cdots n} \frac{S_{ij}\bar{u}_j u_i}{S} = \frac{A}{\lambda} + \frac{B}{\lambda^2} + \cdots.$

PLANE CUBICS WITH A GIVEN QUADRANGLE OF INFLEXIONS.

By B. M. Turner.

That every non-singular cubic has nine points of inflexion, lying in related positions on the curve, is a classical fact in mathematics. Of these points four may be chosen arbitrarily; and when such a quadrangle is fixed, the finding of the positions of the remaining five presents a question worthy of consideration. It appears that all the sets of five combine into a group of fifteen points whose relative positions with respect to the given four depend upon equianharmonic properties; but that the equianharmonic relations follow as a consequence of a combination of harmonic relations and hence, in a number of cases, the points may be determined by linear and quadratic constructions.*

It is also well known that four points of inflexion, no three collinear, impose eight conditions on a cubic and determine it as one of a singly infinite system; but, since only four of the conditions are linear while the other four are of the third degree, the system is not a pencil. It will be shown that the four points determine a system consisting of six pencils and that every two of the six have a fifth point of inflexion in common, that is, through every one of the fifteen points two of the pencils pass, and have consequently an inflexion.

I. Determination and Construction of the Remaining Five Inflexions.

Four points of inflexion of a cubic may be chosen arbitrarily and the conditions imposed by any one of the four are then independent of the conditions imposed by the other three. For a real cubic, however, the imaginary points of inflexion occur in conjugate pairs; hence for such a cubic, if no more than four of the inflexions are involved in the selection, the chosen quadrangle must consist of (1) two pairs of imaginary points, or (2) two real points and one imaginary pair. As a system consisting entirely of imaginary cubics is in itself of little interest, only these two quadrangles supplemented for symmetry by a third with four real vertices will be considered in this discussion. The results for the three cases can be stated in identical terms, as shown in the theorems given in § 3 (pp. 277–8).

* That the whole set of nine points depends only on quadratic constructions was virtually shown by Möbius (Gesammelte Werke, I, p. 437) in the determination of two quadrangles in- and circumscribed to one another. The eight vertices with the addition of the one common diagonal point form the desired set.

§ 1. GIVEN QUADRANGLE—TWO PAIRS OF IMAGINARY POINTS.

Let the two pairs of imaginary points to be taken as points of inflexion for a cubic be given as the intersections of two real lines with a conic (Fig. 1).

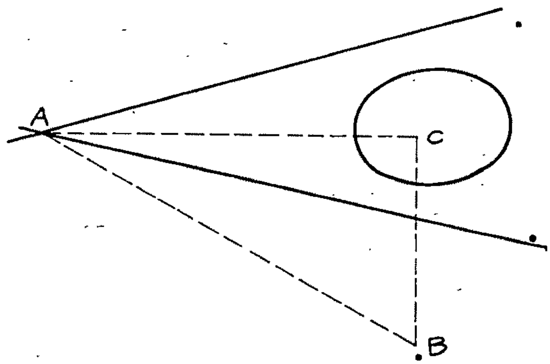

FIG. 1.

Then, as is known, the pencil of conics through the four points has a real common self-polar triangle with A, the intersection of the two given lines, as a vertex and BC, the polar line of A with respect to the given conic, as a side. It is further known that the remaining vertices B, C may be determined by a quadratic construction and hence, since every quadratic construction can be performed by means of the one given conic, the self-polar triangle may be constructed geometrically.*

As the relations to be noted are invariant under projection and any four points forming a quadrangle may be projected into the vertices of any other quadrangle, there is no loss of generality in choosing the points as $(i, \pm 1, \pm 1)$.† Then the self-polar triangle is the triangle of reference, $y \pm z = 0$ are the given lines, and the conic is one of the pencil

$$ax^2 + by^2 + cz^2 = 0$$

where $-a + b + c = 0$.

(i) *Determination of the Five Points.*

Since each component of the three pairs of sides of the quadrangle

$$y^2 - z^2 = 0, \quad z^2 + x^2 = 0, \quad x^2 + y^2 = 0,$$

passes through two of the given points, the six lines are inflexional axes‡

* Two pairs of imaginary points can of course be given by two conics; but if so the determination of the self-polar triangle, also of the real line pair, cannot be accomplished by quadratic constructions.

† Throughout this article the symbol i is used for $\sqrt{-1}$ taken positively. In the cases where an ambiguity enters, it is always preceded by the double sign.

‡ Inflexional axis: a line through three points of inflexion.

for every cubic inflected at the four points. For a real cubic it is known that the real sides of the quadrangle determined by two pairs of its imaginary points of inflexion are sides of the real inflexional triangle, the intersection of the lines forming one pair of imaginary sides of the quadrangle is a point of inflexion, and the intersection of the lines forming the other pair of imaginary sides is the point common to the three real harmonic polars. Hence a real cubic with inflexions at the given four points has the lines through A as sides of the real inflexional triangle, and a fifth inflexion lies either at B or C.

If the fifth point of inflexion is at C, the third side of the real inflexional triangle passes through this point and has an equation of the form $x + \alpha y = 0$, where α is a real number still to be determined. The point B is common to the three real harmonic polars; and the "line of reals,"* being the polar of B with respect to the triangle

$$y \pm z = 0, \qquad x + \alpha y = 0,$$

is $x + 3\alpha y = 0$. Hence since the inflexional axes concurrent with the real harmonic polars ($z \pm ix = 0$) together with the line of reals form a second inflexional triangle, the desired cubic is represented by

$$(z^2 + x^2)(x + 3\alpha y) + \lambda(y^2 - z^2)(x + \alpha y) = 0;$$

and the four remaining points of inflexion, being given by the intersections of $x + 3\alpha y = 0$ with $y^2 - z^2 = 0$ and of $x + \alpha y = 0$ with $z^2 + x^2 = 0$, are

$$(3\alpha,\ -1,\ \pm 1), \qquad (\alpha,\ -1,\ \pm i\alpha).$$

These nine points, namely the four given points, the point C, and the four just found, are points of inflexion for a cubic if, and only if, they satisfy the conditions of collinearity represented by the rows, columns, three right- and three left-hand diagonals of the following scheme:†

$$\begin{array}{lll} (3\alpha,\ -1,\ -1), & (\ i,\ \ 1,\ 1), & (i,\ -1,\ -1), \\ (\ i,\ \ 1,\ -1), & (3\alpha,\ -1,\ 1), & (i,\ -1,\ \ 1), \\ (\ \alpha,\ -1,\ -i\alpha), & (\ \alpha,\ -1,\ i\alpha), & (0,\ \ 0,\ \ 1). \end{array}$$

This requires that $3\alpha^2 = 1$. Accordingly the four points are either

$(\sqrt{3},\ -1,\ \pm 1),\ (1,\ -\sqrt{3},\ \pm i)$ or $(-\sqrt{3},\ -1,\ \pm 1),\ (1,\ \sqrt{3},\ \pm i)$;

and the cubic is a member of one of the two pencils:

$$(1) \qquad (z^2 + x^2)(x + \sqrt{3}y) + \lambda(y^2 - z^2)\left(x + \frac{1}{\sqrt{3}}y\right) = 0,$$

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* The line through the three real points of inflexion.

† Hesse, *Crelle's Journal* (1849), Vol. 38, p. 257; also Clebsch, "Vorlesungen über Geometrie," p. 506.

(2) $$(z^2+x^2)(x-\sqrt{3}y)+\lambda(y^2-z^2)\left(x-\frac{1}{\sqrt{3}}y\right)=0.$$

Similarly if a cubic have B as a fifth point of inflexion, the remaining four inflexions are either
$(\sqrt{3}, \pm 1, -1)$, $(1, \pm i, -\sqrt{3})$ or $(-\sqrt{3}, \pm 1, -1)$, $(1, \pm i, \sqrt{3})$; and the corresponding pencils have equations

(3) $$(y^2-z^2)(z+\sqrt{3}x)+\lambda(x^2+y^2)\left(z+\frac{1}{\sqrt{3}}x\right)=0,$$

(4) $$(y^2-z^2)(z-\sqrt{3}x)+\lambda(x^2+y^2)\left(z-\frac{1}{\sqrt{3}}x\right)=0.$$

For symmetry, A is considered as a fifth point of inflexion, and the set of nine points is completed by either
$(\pm 1, \sqrt{3}, i)$, $(\pm 1, i, -\sqrt{3})$ or $(\pm 1, -\sqrt{3}, i)$, $(\pm 1, i, \sqrt{3})$; but the pencils

(5) $$(x^2+y^2)(y+i\sqrt{3}z)+\lambda(z^2+x^2)\left(y+\frac{i}{\sqrt{3}}z\right)=0,$$

(6) $$(x^2+y^2)(y-i\sqrt{3}z)+\lambda(z^2+x^2)\left(y-\frac{i}{\sqrt{3}}z\right)=0,$$

are imaginary.

These results may be stated in the form of the theorem:

(1) If two pairs of imaginary points of inflexion of a plane cubic are fixed, a real fifth point of inflexion is fixed as one of three, and the complete group of nine is determined as one of six;

or in other words

(1′) Two pairs of imaginary points of inflexion determine a system of cubics consisting of six syzygetic pencils, four real and two imaginary; and every two of the six have a fifth point of inflexion in common.

(ii) *Relative Positions of the Inflexions.*

The curves of the system of cubics have in all nineteen points of inflexion: namely, the four common to all the curves and fifteen others of which every one is common to two of the six pencils. When the four common points are $(i, \pm 1, \pm 1)$, the fifteen are

$$\begin{array}{lll} (1, \ 0, \ 0), & (0, \ 1, \ 0), & (0, \ 0, \ 1), \\ (\sqrt{3}, \pm 1, \pm 1), & (\pm 1, \sqrt{3}, \pm i), & (\pm 1, \pm i, \sqrt{3}). \end{array}$$

These values show that the points determine certain quadranges. The lines

$$y \pm z = 0, \qquad z \pm ix = 0, \qquad x \pm iy = 0$$

are the sides of the chosen quadrangle; while the sides of the determined quadrangles are

$$y \pm z = 0, \qquad z \pm \frac{1}{\sqrt{3}}x = 0, \qquad x \pm \sqrt{3}y = 0;$$

$$y \pm i\sqrt{3}z = 0, \qquad z \pm ix = 0, \qquad x \pm \frac{1}{\sqrt{3}}y = 0;$$

$$y \pm \frac{1}{\sqrt{3}}z = 0, \qquad z \pm \sqrt{3}x = 0, \qquad x \pm iy = 0;$$

Hence follows the theorem:

(2) The remaining points of inflexion of the cubics with two common pairs of imaginary inflexions form a group of fifteen, consisting of the three diagonal points of the common quadrangle and the vertices of three other quadrangles with the same diagonal points, each of the three new quadrangles having one pair of sides in common with the original quadrangle.

The sides of the quadrangles and their common diagonal triangle are further related. The six real lines through C considered as three pairs

$$x = 0,\ y = 0;\ x + \sqrt{3}y = 0,\ x - \frac{1}{\sqrt{3}}y = 0;\ x - \sqrt{3}y = 0,\ x + \frac{1}{\sqrt{3}}y = 0$$

form a pencil in elliptic involution with

$$x \pm iy = 0$$

as double lines. The lines of any one of the three pairs are harmonic with respect to the first lines of the other two pairs, and also with respect to the last lines. Furthermore either triad of lines

$$x = 0, \qquad x \pm \sqrt{3}y = 0 \qquad \text{or} \qquad y = 0, \qquad x \pm \frac{1}{\sqrt{3}}y = 0,$$

together with either double line, forms an equianharmonic system.* Thus the lines

$$x \pm \sqrt{3}y = 0, \qquad x \pm \frac{1}{\sqrt{3}}y = 0$$

satisfy a combination of harmonic and equianharmonic relations with respect to $x = 0$, $y = 0$, $x \pm iy = 0$. The equianharmonic properties, however, are simply a consequence of harmonic properties; for if we write

$$x = 0, \quad y = 0; \quad x + \beta y = 0, \quad x - \alpha y = 0; \quad x - \beta y = 0, \quad x + \alpha y = 0,$$

* Consider $x = 0$, $x \pm \sqrt{3}y = 0$, $x + iy = 0$. Let $X = x + \sqrt{3}y$, $Y = x - \sqrt{3}y$; then $x = 0$, $x + iy = 0$ are transformed respectively into $X + Y = 0$, $X - \omega Y = 0$; and the cross-ratio of the four lines $X = 0$, $Y = 0$, $X + Y = 0$, $X - \omega Y = 0$ is $-\omega^2$, where $\omega^3 = 1$. Similarly for the other combinations.

and impose the condition that $x = 0$, $x + \beta y = 0$ be harmonic with respect to $x - \beta y = 0$, $x + \alpha y = 0$, we find $\beta = 3\alpha$; and the condition that $x + 3\alpha y = 0$, $x - \alpha y = 0$ be harmonic with respect to $x \pm iy = 0$ shows that $3\alpha^2 = 1$.

Similar statements hold for the lines through B; and also for the set through A, except that in this case the involution is hyperbolic. Consequently the sides of the three quadrangles, and hence the vertices, are uniquely determined from the sides of the base quadrangle and its diagonal triangle by means of harmonic properties.

The equianharmonic properties furnish an analytic means of determining the points, and also serve to show the relative positions of the three quadrangles with respect to the four given points. The points equianharmonic to $(1, 0, 0)$ with respect to $(i, 1, 1)$, $(i, -1, 1)$ are $(\sqrt{3}, 1, 1)$ and $(3, -1, -1)$; and those equianharmonic with respect to $(i, -1, 1)$, $(i, 1, -1)$ are $(\sqrt{3}, -1, 1)$, $(\sqrt{3}, 1, -1)$. Thus the vertices of the real quadrangle $(\sqrt{3}, \pm 1, \pm 1)$ are the points on the lines $y \pm z = 0$ equianharmonic to $(1, 0, 0)$ with respect to $(i, \pm 1, \pm 1)$. Similarly $(\pm 1, \sqrt{3}, \pm i)$ are the points on $z \pm ix = 0$ and $(1, \pm i, \sqrt{3})$ the points on $x \pm iy = 0$ equianharmonic to $(0, 1, 0)$ and $(0, 0, 1)$ with respect to the four given points. These results are expressed in the following theorem:

(3) The fifteen other possible points of inflexion of a cubic with two fixed imaginary pairs are the three diagonal points of the fixed quadrangle, and the two points on every one of the six sides of the quadrangle equianharmonic to the diagonal point with respect to the two fixed points on that side.

The vertices of the three determined quadrangles may also be obtained analytically as the intersections of three conics. Three pairs of lines

$$y \pm iz = 0, \qquad z \pm x = 0, \qquad x \pm y = 0$$

are uniquely determined as being harmonic both with respect to the sides of the given quadrangle $(i, \pm 1, \pm 1)$ and with respect to the sides of the diagonal triangle. Three conics having respectively these pairs of lines as tangents, namely,

$$2x^2 - y^2 - z^2 = 0, \qquad x^2 - 2y^2 - z^2 = 0, \qquad x^2 - y^2 - 2z^2 = 0$$

pass, taken in the same order, through the pairs of quadrangles

$$\begin{array}{ll} (\pm 1, \sqrt{3}, \pm i), & (\pm 1, \pm i, \sqrt{3}); \\ (\pm 1, \pm i, \sqrt{3}), & (\sqrt{3}, \pm 1, \pm 1); \\ (\sqrt{3}, \pm 1, \pm 1), & (\pm 1, \sqrt{3}, \pm i). \end{array}$$

Further any one of the three conics passes through the four intersections

of the pairs of tangents given for the other two. Thus any one of these conics is uniquely determined by four points and two tangents, the determining elements being fixed by means of harmonic properties with respect to the given quadrangle. In turn the three conics uniquely determine the three quadrangles

$$(\sqrt{3}, \pm 1, \pm 1), \qquad (\pm 1, \sqrt{3}, \pm i), \qquad (\pm 1, \pm i, \sqrt{3}).$$

Accordingly the fifteen other possible points of inflexion of a cubic with two fixed imaginary pairs are the three diagonal points of the fixed quadrangle, and the twelve intersections of three conics uniquely determined analytically by the quadrangle.

(iii) *Actual Construction of the Points.*

It has been noted that when two pairs of imaginary points are given as the intersections of two real lines with a conic, there follows a quadratic construction for the triangle self-polar for the pencil of conics through the four points.* It will now be shown that with the help of the triangle, the fifteen other possible points of inflexion of a cubic inflected at the two pairs of imaginary points may be determined by a series of constructions of which one only is quadratic, the rest linear.

As A is the intersection of the two given lines, the given conic meets BC certainly and either AB or CA in real points. Let it meet CA and call the points D, D'. Denote the intersections of BC with the given lines by E, E'.

The given points being $(i, \pm 1, \pm 1)$, the object is to construct the lines

$$x + \sqrt{3}y = 0, \qquad x \pm \frac{1}{\sqrt{3}} y = 0, \qquad z \pm \sqrt{3}x = 0, \qquad z \pm \frac{1}{\sqrt{3}} x = 0.$$

The conic of the pencil through the four given points that meets $y = 0$ on the lines $z \pm \sqrt{3}x = 0$ is $3x^2 + by^2 - z^2 = 0$, with the condition that $-3 + b - 1 = 0$, that is, the conic

$$3x^2 + 4y^2 - z^2 = 0.$$

This conic meets $x = 0$ where $4y - z = 0$, hence the first step requires the construction of the points of intersection of the lines $2y - z = 0$, $2y + z = 0$ with BC. Since

$$\begin{array}{llll} 2y - z = 0, & z = 0; & y - z = 0, & y = 0, \\ 2y + z = 0, & z = 0; & y + z = 0, & y = 0 \end{array}$$

are two sets of harmonic lines, these points P_1, P_2 may be constructed

* See page 262.

linearly. Draw P_1D intersecting the given conic in a second point D'' and the given lines in F, F'. As the conics of a pencil cut any line in involution, P_3 the conjugate to P_1 in the involution $(D''D, FF')$ is another point on the conic

$$3x^2 + 4y^2 - z^2 = 0.$$

This conic is a member of a second pencil through two points P_1 (AP_1 being a tangent at a given point), the point P_2,* and the point P_3. The pairs of lines

$$AP_1, \quad P_2P_3; \qquad P_1P_2, \quad P_1P_3$$

are two other conics of the second pencil: hence this pencil cuts out on CA the involution (AV, CD), where V is the intersection of P_2P_3 with CA. The involution cut out on CA by the first pencil (the pencil through the four given points) is defined by its two double points, and may be expressed as (A^2, C^2). It follows that Q, Q', the common points of the two involutions

$$(AV, CD), \qquad (A^2, C^2),$$

found by means of the given conic (the one quadratic construction), are the intersections of $3x^2 + 4y^2 - z^2 = 0$ with $y = 0$. Hence BQ, BQ' are the desired lines

$$z \pm \sqrt{3}x = 0.$$

Since

$$z - \frac{1}{\sqrt{3}}x = 0, \qquad z + \sqrt{3}x = 0; \qquad z = 0, \qquad z - \sqrt{3}x = 0,$$

$$z + \frac{1}{\sqrt{3}}x = 0, \qquad z - \sqrt{3}x = 0; \qquad z = 0, \qquad z + \sqrt{3}x = 0$$

are two sets of harmonic lines, two other of the desired lines,

$$z \pm \frac{1}{\sqrt{3}}x = 0,$$

may be constructed linearly: call them BK, BK'. The lines that join the intersections of $z \pm \sqrt{3}x = 0$, $z \pm \dfrac{1}{\sqrt{3}}x = 0$ with $y \pm z = 0$ to C are

$$x \pm \sqrt{3}y = 0, \qquad x \pm \frac{1}{\sqrt{3}}y = 0.$$

Hence the complete construction (Fig. 2) may be stated as follows:

Construct P_1, P_2 the harmonic conjugates of B with respect to E, C and E', C. Draw P_1D intersecting the given conic in a second

* The conic $3x^2 + 4y^2 - z^2 = 0$ also has AP_2 for a tangent, but the use of two points P_1 and two points P_2 would give an illusory construction.

point D'' and the line-pair in F, F'. Construct P_3 the conjugate to P_1 in the involution $(D''D, FF')$. Draw P_2P_3 intersecting CA in V. Determine Q, Q' the common points * for the two involutions $(AV,$

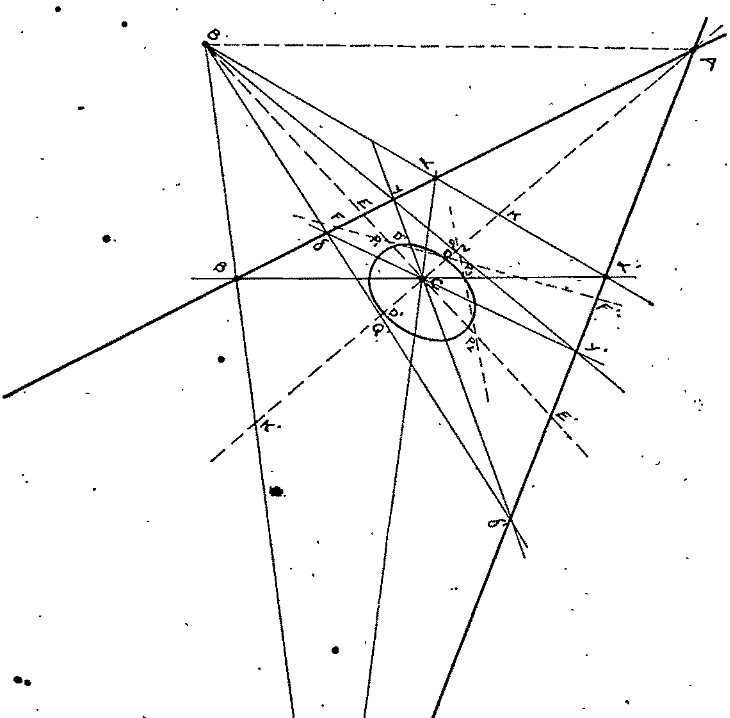

FIG. 2.

$CD)$, (A^2, C^2). Construct K, K' the harmonic conjugates of Q' with respect to A, Q and of Q with respect to A, Q'. Draw BK, BK', BQ, BQ' intersecting the given line-pair in α, α'; β, β'; γ, γ'; δ, δ'. Draw $\alpha\beta'$, $\alpha'\beta$, $\gamma\delta'$, $\gamma'\delta$.

The seven real points of inflexion are

$$A,\ B,\ C,\ \alpha,\ \alpha',\ \beta,\ \beta'.$$

* That Q, Q' are real is shown by the analytical discussion.

The eight imaginary points lie by pairs on the lines

$$\gamma\gamma', \ \delta\delta', \ \gamma\delta', \ \gamma'\delta,$$

where they are met by the two pairs of imaginary sides of the given quadrangle.

(iv) *Symmetrical Constructions.*

When the two pairs of imaginary inflexions are given as the intersections of two equal hyperbolas with the same pair of axes, the construction (Fig. 3) of the fifteen points is unique and furnishes an illustration of the three conics which intersect in the vertices of the determined quadrangles.

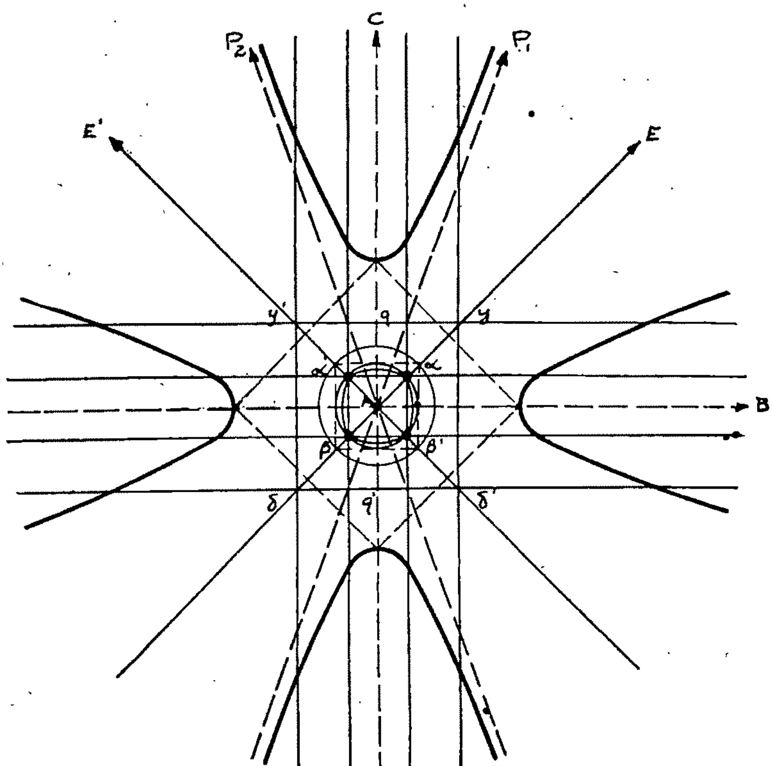

FIG. 3.

The hypothesis gives the axes and line at infinity, that is, the self-polar triangle; and (keeping the lettering of the preceding construction) the point A is the common center of the two hyperbolas. Draw the four other lines joining the vertices of the hyperbolas; then AE, AE', the real line-pair through the four given points,* bisect these lines. The lines AP_1, AP_2

* The equality of the hyperbolas accounts for the construction of these lines, in general not possible by quadratic construction when the two pairs of imaginary points are given by two conics.—See note, page 5.

are harmonic to AB with respect to AE, CA and AE', CA; and the points Q, Q' are determined as the vertices of the hyperbola through the four given points having AP_1, AP_2 as asymptotes.* Then the lines through Q, Q' parallel to AE together with the lines joining their intersections with the real line-pair are $\gamma\gamma'$, $\delta\delta'$, $\gamma\delta'$, $\gamma'\delta$; and by means of the harmonic relations between these lines and the axes AB, CA the lines $\alpha\alpha'$, $\beta\beta'$, $\alpha\beta'$, $\alpha'\beta$ may be constructed.

The above construction determines the vertices of the three quadrangles as the intersections of lines. To show them as the intersections of conics, let the two given hyperbolas be members of the pencil $ax^2 + by^2 + cz^2 = 0$, where $-a + b + c = 0$, when (1) $x = 0$ is the line at infinity and (2) $y = 0$, $z = 0$ and $y + z = 0$, $y - z = 0$ are two pairs of perpendicular lines. Then the three conics intersecting in the vertices of the determined quadrangles are two equal, symmetrically placed ellipses,

$$2y^2 + z^2 = x^2, \qquad y^2 + 2z^2 = x^2,$$

and the circle

$$y^2 + z^2 = 2x^2,$$

all three concentric with the hyperbolas.

Accordingly construct the two equal, symmetrically placed ellipses through α, α', β, β'; and pass a circle through the four finite intersections of the tangents to the ellipses at their vertices. Then the fifteen other possible points of inflexion of a cubic inflected at the four imaginary intersections of the two hyperbolas are the common center, the two points at infinity on the axes, the four real intersections of the two ellipses, and the eight imaginary intersections of the circle with the ellipses.

Another symmetrical construction is obtained by projecting one pair of the given points into the circular points. Then, the pencil of conics through the two pairs of imaginary points is a system of coaxial circles, the real line-pair consists of the radical axis and the line at infinity, and two vertices of the self-polar triangle are the limiting points of the system while the third vertex is at infinity on the radical axis. A pair of circles, each having one limiting point as a center and passing through the other limiting point, intersect on the radical axis in two vertices of the quadrangle of real points. A second pair of circles, having these vertices as centers and passing through the limiting points, determine four other points (z) on the first pair. The lines joining these four points to the limiting points pass through the remaining possible points of inflexion of cubics inflected at the

* Draw a line parallel to AP_1 intersecting the real line-pair and the hyperbola with AB as transverse axis. The center of the involution determined by the two pairs of points of intersection is a point on the hyperbola having AP_1 and AP_2 as asymptotes; and it is known that a hyperbola can be constructed when the asymptotes and one point on the curve are given.

four given imaginary points. This gives a unique construction when the two pairs of imaginary points are taken as the intersections of two circles. See (Fig. 4), where to complete the symmetry the two given circles are drawn equal.

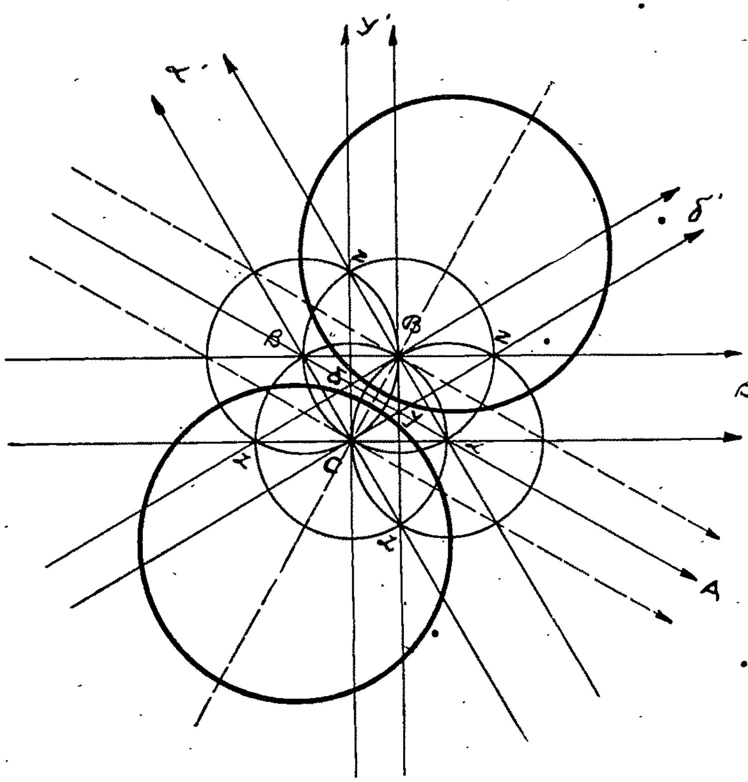

FIG. 4.

For the proof of the construction project $(\pm i, 1, 1)$ into the circular points and change from homogeneous to Cartesian coördinates. The equation of the system of coaxial circles is then

$$x^2 + y^2 - 2\lambda y + \lambda = 0,$$

with $2y - 1 = 0$ as the radical axis and $(0, 0)$, $(0, 1)$ as the limiting points. The two circles each having one limiting point as center and passing through the other are

$$x^2 + y^2 = 1, \qquad x^2 + (y - 1)^2 = 1;$$

and these circles intersect on the radical axis in $(\pm \frac{1}{2}\sqrt{3}, \frac{1}{2})$, or $(\pm \sqrt{3}, 1, -1)$ in the homogeneous coördinates given by $z = y - 1$. The second pair of circles having these two points as centers and passing through the limiting points, namely,

$$(x - \tfrac{1}{2}\sqrt{3})^2 + (y - \tfrac{1}{2})^2 = 1, \qquad (x + \tfrac{1}{2}\sqrt{3})^2 + (y - \tfrac{1}{2})^2 = 1,$$

determine on the first pair the points

$$(\tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}), \quad (-\tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}), \quad (\tfrac{1}{2}\sqrt{3}, \tfrac{3}{2}), \quad (-\tfrac{1}{2}\sqrt{3}, \tfrac{3}{2}),$$

or

$$(\sqrt{3}, -1, -3), \quad (-\sqrt{3}, -1, -3), \quad (\sqrt{3}, 3, 1), \quad (-\sqrt{3}, 3, 1);$$

and the lines joining these four points to the limiting points are

$$x \pm \sqrt{3}y = 0, \quad x \pm \frac{1}{\sqrt{3}}y = 0, \quad z \pm \sqrt{3}x = 0, \quad z \pm \frac{1}{\sqrt{3}}x = 0.$$

§ 2. GIVEN QUADRANGLE—TWO REAL AND A PAIR OF IMAGINARY POINTS.

Let four points, two real and one imaginary pair, to be taken as points of inflexion for a cubic be determined geometrically as the intersections of two real lines with a conic (Fig. 5). It is then known that the common

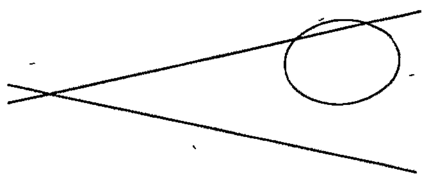

FIG. 5.

self-polar triangle for the pencil of conics through the four points has one real vertex, the intersection of the two given lines, and one real side, the polar of the real vertex with respect to the given conic; while the two remaining vertices and sides of the triangle are imaginary.

The study of the cubics with two real and a pair of imaginary inflexions fixed is correlated with the preceding study of the cubics with two fixed pairs of imaginary inflexions, by choosing the four points as $(\sqrt{3}, 1, \pm 1)$, $(1, \sqrt{3}, \pm i)$.*

(i) *Determination of the Five Points.*

The procedure followed in the case of the two pairs of imaginary points shows that the cubics with the four given inflexions have six common inflexional axes, namely, the three pairs of side of the given quadrangle,

$$x - \sqrt{3}y = 0, \qquad x - \frac{1}{\sqrt{3}}y = 0,$$

$$i\omega^2 x + \omega y \pm z = 0, \qquad -i\omega x + \omega^2 y \pm z = 0, \qquad (\omega^3 = 1).$$

Also as before a fifth point of inflexion is one of three: the real point A $(0, 0, 1)$ or either of the pair of imaginary points B $(i, \omega, 0)$, C $(-i, \omega^2, 0)$.

* See page 263.

Consider first a cubic with the real point A as a fifth point of inflexion. The cubic has two imaginary inflexional axes through this point; and the equation is consequently of the form

$$(i\omega^2 x + \omega y + z)(i\omega^2 x + \omega y - z)(x + \alpha y) + \lambda(-i\omega x + \omega^2 y + z)(-i\omega x + \omega^2 y - z)(x + \beta y) = 0,$$

where α, β are a pair of complex numbers. Accordingly the remaining four inflexions are at

$$(\alpha, -1, i\omega\alpha + \omega^2), \qquad (\alpha, -1, -i\omega\alpha - \omega^2),$$
$$(\beta, -1, i\omega^2\beta - \omega), \qquad (\beta, -1, -i\omega^2\beta + \omega),$$

where, in order to satisfy the conditions of collinearity imposed on every group of inflexions of a non-singular cubic, that is, the conditions represented by the scheme

$$\begin{matrix} (\sqrt{3}, 1, 1), & (\beta, -1, -i\omega^2\beta + \omega), & (\alpha, -1, i\omega\alpha + \omega^2), \\ (\beta, -1, i\omega^2\beta - \omega), & (\sqrt{3}, 1, -1), & (\alpha, -1, -i\omega\alpha - \omega^2), \\ (1, \sqrt{3}, i), & (1, \sqrt{3}, -i), & (0, 0, 1) \end{matrix}$$

either $\alpha = -i$, $\beta = i$, or $\alpha = \frac{1}{7}(-i - 4\sqrt{3})$, $\beta = \frac{1}{7}(i - 4\sqrt{3})$.

If $\alpha = -i$, $\beta = i$, the four points are $(i, \pm 1, \pm 1)$ in agreement with the results in the preceding case. If $\alpha = \frac{1}{7}(-i - 4\sqrt{3})$, $\beta = \frac{1}{7}(i - 4\sqrt{3})$, a second set of four points is obtained; but since the computations involved are complicated it is advantageous to apply a linear transformation by which the original four points become

$$(0, 1, -1), \qquad (-1, 0, 1), \qquad (1, -\omega, 0), \qquad (1, -\omega^2, 0).$$

Then the fifth point is $(1, -1, 0)$; the remaining four are either

$$(0, 1, -\omega), \qquad (0, 1, -\omega^2), \qquad (-\omega, 0, 1), \qquad (-\omega^2, 0, 1),$$

or

$$(-1, \omega^2 - 1, 1), \ (-1, \omega - 1, 1), \ (\omega^2 - 1, -1, 1), \ (\omega - 1, -1, 1):$$

and the corresponding pencils of cubics may be written as

$$x^3 + y^3 + z^3 + \lambda xyz = 0,$$
$$x^3 + y^3 + z^3 + 3z(x^2 + y^2) + 3z^2(x + y) + \lambda z(z + x)(y + z) = 0.$$

Similarly if either B or C is the fifth point of inflexion, there are two distinct pencils of cubics; but, since the four pencils thus determined are imaginary and consequently of interest in this discussion only to give symmetry to the results, their equations and the coördinates of their remaining points of inflexion are omitted.

The cubics of the three pairs of pencils have in all only nineteen points of inflexion, and these have the same relative positions as the nineteen for the three pairs of pencils determined by two pairs of imaginary inflexions. Hence, it follows that, with the proper interchanging of the words real and imaginary, the theorems stated on pages 264, 265, and 266 hold for the cubics with two real and an imaginary pair of fixed inflexions. The constructions, however, because of the great number of imaginary elements involved become almost entirely theoretical.

(ii) *Related Quartic Curves.*

The equations

$$P_1 \equiv x^3 + y^3 + z^3 + \lambda xyz = 0,$$
$$P_2 \equiv x^3 + y^3 + z^3 + 3z(x^2 + y^2) + 3z^2(x + y) + \lambda z(z + x)(x + y) = 0$$

represent two pencils of cubics having in common three real and a pair of imaginary inflexions. Every value of λ determines a definite curve in each pencil, and the elimination of λ between the two equations gives

$$z^2[(x^3 + y^3 + z^3)(x + y + z) - 3xy(x^2 + y^2) - 3xyz(x + y)] = 0$$

as the locus of the intersections of the two curves. Three of the fixed intersections, namely, one real and the given pair of imaginary inflexions, are on $z = 0$; hence this line is a part of the locus only because of the two curves of which it forms a part. The remaining two fixed inflexions and the four variable intersections lie on the quartic

$$(x^3 + y^3 + z^3)(x + y + z) - 3xy(x^2 + y^2) - 3xyz(x + y) = 0.$$

The remaining two fixed inflexions are $(0, 1, -1)$, $(-1, 0, 1)$, where the quartic has nodes with tangents

$$\pm ix + y + z = 0, \qquad x \pm iy + z = 0;$$

and these lines are the inflexional tangents common to the two cubics considered when $\lambda = \pm 3i$. Thus the binodal quartic is the locus of the four variable interesections of the two curves of P_1 and P_2 having the same parametric value.

The two sets of four points

$$(0,\ 1,\ -\omega), \quad (0,\ 1,\ -\omega^2), \quad (-\omega,\ 0,\ 1), \quad (-\omega^2,\ 0,\ 1);$$
$$(-1,\ \omega^2 - 1,\ 1), \quad (-1,\ \omega - 1,\ 1), \quad (\omega^2 - 1,\ -1,\ 1), \quad (\omega - 1,\ -1,\ 1),$$

completing the inflexional groups of P_1 and P_2 are on the quartic; and these together with the two double points $(0, 1, -1)$, $(-1, 0, 1)$ are the complete intersection of two quartics

$$(\omega x + y + z)(\omega^2 x + y + z)(x + \omega y + z)(x + \omega^2 y + z) = 0,$$
$$xy(z + x)(x + y) = 0,$$

each composed of two pairs of lines, one pair through each of the double points. Accordingly

$$(\omega x + y + z)(\omega^2 x + y + z)(x + \omega y + z)(x + \omega^2 y + z) + \mu xy(z + x)(x + y) = 0$$

is a pencil of binodal quartics through the eight points and two nodes. If the second pencil of cubics is written

$$P_2 \equiv x^3 + y^3 + z^3 + 3z(x^2 + y^2) + 3z^2(x + y) + (\lambda - n)z(z + x)(y + z) = 0,$$

where n has any real value, a different quartic of the pencil corresponds to each chosen value of n, that is, the manner of writing the equations of P_1 and P_2 can be so varied that every quartic of the pencil may be found as the locus of the four variable intersections of the two cubics with the same parametric value. The two reducible quartics corresponding to μ infinite or zero are obtained respectively when $n = \infty$ or $n = 3$. If $n = \infty$, P_2 breaks up into three lines. If $n = 3$, the two cubics have the same tangents at the two common imaginary inflexions $(1, -\omega, 0)$, $(1, -\omega^2, 0)$ for every value of λ; and all their intersections lie at the five given points except when the cubics have a common linear factor. In the study of non-singular cubics these two cases are excluded, and hence the following result can be stated:

There are two, and only two, pencils of cubics having in common three real and a pair of imaginary inflexions; and the locus of the four variable intersections of the two corresponding curves is a binodal quartic passing through the remaining inflexions of the two pencils, the two nodes being at the two real inflexions not collinear with the two common imaginary inflexions.

In further consideration of the geometry on the pairs of cubics and the resulting quartic curve it may be noted that three of the nine intersections of the two cubics lie on a line, hence the remaining six, the six on the quartic, lie on a conic. The equation

$$P_1 - P_2 \equiv 3z[x^2 + y^2 + z(x + y) + \frac{\lambda}{3}z(x + y + z) = 0$$

represents a pencil of cubics consisting of the line and a pencil of conics. Every value of λ determines a curve of P_1 and P_2 and a conic through their six intersections on the quartic. The pencil of conics passes through the two nodes $(0, 1, -1)$, $(-1, 0, 1)$, and the two points $(1, \pm i, 0)$ where the line $z = 0$ is intersected by the nodal tangents

$$\pm ix + y + z = 0, \qquad x \pm iy + z = 0.$$

Thus the four variable intersections of the two cubics are cut out on the

binodal quartic by a pencil of concis through the two nodes and the two intersections of the nodal tangents collinear with the two common imaginary inflexions.

When the points $(1, \pm i, 0)$ are projected into the circular points, the special case arises where the four variable intersections lie on a circle through two of the common real points of inflexion. It may also be noticed that, since the four variable intersections of the two cubics and the two nodes of the quartic lie on a conic, the four variable intersections subtend at the two nodal points pencils of lines with the same cross-ratio.

§ 3. GENERAL CONCLUSIONS.

For symmetry the cubics with a fixed quadrangle of real points of inflexion are considered, although every such cubic is imaginary. Choose the four points $(\sqrt{3}, \pm 1, \pm 1)$. Then the six fixed inflexional axes are

$$y \pm z = 0, \qquad z \pm \frac{1}{\sqrt{3}} x = 0, \qquad x \pm \sqrt{3} y = 0;$$

and a fifth point of inflexion is any one of the three

$$(1, 0, 0), \qquad (0, 1, 0), \qquad (0, 0, 1).$$

It follows that a cubic of the system with an inflexion at $(1, 0, 0)$ is a member of one of the pencils

$$(x^2 - 3y^2)(y + i\sqrt{3}z) + \lambda(z^2 - \tfrac{1}{3}x^2)\left(y + \frac{i}{\sqrt{3}} z\right) = 0,$$

$$(x^2 - 3y^2)(y - i\sqrt{3}z) + \lambda(z^2 - \tfrac{1}{3}x^2)\left(y - \frac{i}{\sqrt{3}} z\right) = 0;$$

and similar results hold with respect to $(0, 1, 0)$ and $(0, 0, 1)$. Furthermore the six pencils have nineteen points of inflexion associated as in the two preceding cases. Hence the theorems already stated (pp. 264, 265, 266) are applicable to this case also, that is, for cubics with any fixed quadrangle of inflexions* the following theorems hold:

† (1) If four points of inflexion of a plane cubic, no three collinear, are fixed, a fifth point of inflexion is fixed as one of three, and the complete group of nine is determined as one of six; or in other words,

(1′) A quadrangle of inflexions determines a system of cubics consisting of six syzygetic pencils, and every two of the six have a fifth point of inflexion in common.

* Provided, as stated on page 261, that if imaginary the points enter by conjugate pairs.

† A. Wiman, Nyt Tiddskrift for Matematic (1894); also W. Burnside, *Proc. London Math. Soc.* (1906–07).

(2) The remaining points of inflexion of the cubics with a common quadrangle of inflexions form a group of fifteen, consisting of the three diagonal points of the common quadrangle and the vertices of three other quadrangles with the same diagonal points, each of the three new quadrangles having one pair of sides in common with the original quadrangle.

(3) The fifteen other possible points of inflexion of a cubic with a fixed quadrangle of inflexions are the three diagonal points of the fixed quadrangle, and the two points on every one of the six sides of the quadrangle equianharmonic to the diagonal point with respect to the two fixed points on that side.

NORMAL TERNARY CONTINUED FRACTION EXPANSIONS FOR THE CUBE ROOTS OF INTEGERS.*

By P. H. Daus.

Introduction.

The problem of finding rational approximations to a cubic irrationality by means of a periodic expansion was first attempted and partially solved by Jacobi† in 1868. Jacobi devised an extension of continued fractions, which enabled him to determine rational approximations to the mutual ratios of three numbers of such a nature that any approximation (A_n, B_n, C_n) could be expressed in terms of the three preceding ones by means of the recursion formulæ,

$$(A) \qquad \begin{aligned} A_n &= p_n A_{n-1} + q_n A_{n-2} + A_{n-3}, \\ B_n &= p_n B_{n-1} + q_n B_{n-2} + B_{n-3}, \\ C_n &= p_n C_{n-1} + q_n C_{n-2} + C_{n-3}. \end{aligned}$$

He further showed that if the three numbers be taken $1 : \theta : a + b\theta + c\theta^2$, where θ is the real root of a rational cubic equation of negative discriminant, that is, one with one real root, the expansion may become periodic, and in actual numerical cases does so. It has been proved by Bachman,‡ however, that this periodicity, using Jacobi's method of selecting (p_n, q_n), can not exist unless a certain limiting condition be satisfied. Berwick§ has obtained periodic expansions for cubic irrationalities, but his algorithm differs from Jacobi's, and has the disadvantage that the transformations involved are not necessarily unimodular and may be singular. Lehmer,¶ instead of starting with a cubic irrationality, started with a periodic expansion and found associated with it a definite cubic irrationality.

Closely allied to this problem, in the case when $\theta = \sqrt[3]{D}$, is the solution of the Pellian cubic

$$(B) \qquad F(x, y, z) = x^3 + Dy^3 + D^2z^3 - 3Dxyz = m.$$

The value of $F(x, y, z)$ can be expressed in several other ways which we

* Since this article was submitted for publication, the results have been extended to the roots of the more general cubic equation $x^3 + px^2 + qx + r = 0$. These results will be the subject of a later article.

† Jacobi, C. G. J., *Ges. Werke*, VI (385–426).

‡ Bachman, P., *Crelle*, Vol. 75 (1873) (25–34).

§ Berwick, W. E. H., *Proc. London Math. Soc.* (Ser. 2), Vol. 12 (1912) (393–429).

¶ Lehmer, D. N., *Proc. Nat. Acad. of Sciences*, Vol. 4, Dec., 1918 (360–364).

shall find useful. As a determinant

$$(C) \qquad F(x, y, z) = \begin{vmatrix} x & Dz & Dy \\ y & x & Dz \\ z & y & x \end{vmatrix}.$$

Or as a product of three conjugate factors

$$(D) \quad F(x, y, z) = (x + y\theta + z\theta^2)(x + \omega\theta y + \omega^2\theta^2 z)(x + \omega^2\theta y + \omega\theta^2 z),$$

where ω is an imaginary cube root of unity. Dickson* in his History of the Theory of Numbers describes briefly the work done along this line. Tables of solutions have been computed by Meissel† for $D < 82$ (although he did not publish his table) and an incomplete table by Wolfe‡ for $D < 100$. This latter table has been used in computations made for this paper.

The object of this paper is to show that by a suitable choice of (p_n, q_n), Jacobi's method may be modified, and that the modified expansions bear close analogies to ordinary continued fractions. This leads to a definition of normal ternary continued fractions and an attempt to compute a table of such expansions for the cube roots of integers.

If u_1, v_1, w_1 be any three numbers, we define a ternary continued fraction expansion for them by the equations

$$u_{n+1} = v_n - p_n u_n, \qquad v_{n+1} = w_n - q_n u_n, \qquad w_{n+1} = u_n,$$

where (u_n, v_n, w_n) is called the nth complete quotient set, (p_n, q_n) the nth partial quotient set and (A_n, B_n, C_n), as defined by equations (A) of the introduction, the nth convergent set to the ternary continued fraction

$$\left(\frac{v_1}{u_1}, \frac{w_1}{u_1}\right) = (p_1, q_1;\ p_2, q_2;\ p_3, q_3;\ \cdots).$$

The following set of relations connecting the first and general complete quotient sets is fundamental.

THEOREM I. *If u_1, v_1, w_1, any three numbers, be expanded into a ternary continued fraction, and if $\sigma_{1,n} = v_n/u_n$, $\sigma_{2,n} = w_n/u_n$, and (A_n, B_n, C_n) be the nth convergent set, then*

$$\sigma_{1,1} = \frac{B_n\sigma_{2,n+1} + B_{n-1}\sigma_{1,n+1} + B_{n-2}}{A_n\sigma_{2,n+1} + A_{n-1}\sigma_{1,n+1} + A_{n-2}}$$

and

$$\sigma_{2,1} = \frac{C_n\sigma_{2,n+1} + C_{n-1}\sigma_{1,n+1} + C_{n-2}}{A_n\sigma_{2,n+1} + A_{n-1}\sigma_{1,n+1} + A_{n-2}}.$$

* Dickson, L. E., Vol. II (593–595).

† Meissel, E., Program, Kiel, 1891.

‡ Wolfe, C. (work done at the Univ. of Calif.; not yet published).

The defining equations may be written by combining them,

$$
(1)\qquad \begin{aligned}
u_n &= && && w_{n+1}\\
v_n &= u_{n+1} && && + p_n w_{n+1}\\
w_n &= && v_{n+1} && + q_n w_{n+1}
\end{aligned}
$$

defining a unimodular collineation, and we can write

$$
(2)\qquad \begin{aligned}
u_1 &= A_{n-2}u_{n+1} + A_{n-1}v_{n+1} + A_n w_{n+1}\\
v_1 &= B_{n-2}u_{n+1} + B_{n-1}v_{n+1} + B_n w_{n+1}\\
w_1 &= C_{n-2}u_{n+1} + C_{n-1}v_{n+1} + C_n w_{n+1}
\end{aligned}
$$

where

$$
\begin{vmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{vmatrix} = \prod_1^n \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & p_n \\ 0 & 1 & q_n \end{vmatrix}.
$$

Now

$$
\begin{vmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{vmatrix} \times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & p_{n+1} \\ 0 & 1 & q_{n+1} \end{vmatrix} = \begin{vmatrix} A_{n-1} & A_n & A_{n-2}+p_{n+1}A_{n-1}+q_{n+1}A_n \\ B_{n-1} & B_n & B_{n-2}+p_{n+1}B_{n-1}+q_{n+1}B_n \\ C_{n-1} & C_n & C_{n-2}+p_{n+1}C_{n-1}+q_{n+1}C_n \end{vmatrix},
$$

from which we have the recursion formulæ for convergent sets

$$
(3)\qquad \begin{aligned}
A_n &= p_n A_{n-1} + q_n A_{n-2} + A_{n-3}\\
B_n &= p_n B_{n-1} + q_n B_{n-2} + B_{n-3}\\
C_n &= p_n C_{n-1} + q_n C_{n-2} + C_{n-3}
\end{aligned}
$$

with the initial conditions

$$
\begin{vmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.
$$

In actual calculations we interchange rows and columns and compute successive convergents by the recursion formulæ (3). From equations (2) by division to make the expressions non-homogeneous, we have the equations stated in the theorem.

By solving equations (2) for u_{n+1}, v_{n+1}, and w_{n+1} we have

$$
(4)\qquad \begin{aligned}
u_{n+1} &= c_{n-2}w_1 + b_{n-2}v_1 + a_{n-2}u_1,\\
v_{n+1} &= c_{n-1}w_1 + b_{n-1}v_1 + a_{n-1}u_1,\\
w_{n+1} &= c_n\ \ w_1 + b_n\ \ v_1 + a_n\ \ u_1,
\end{aligned}
$$

where a_n, b_{n-1}, etc., are the cofactors of the corresponding elements of the determinant in (2). Dividing v_{n+1} and w_{n+1} respectively by u_{n+1}, we obtain as a

COROLLARY. *Under the conditions of theorem I*

$$
(5)\qquad \begin{aligned}
\sigma_{2,\,n+1} &= \frac{c_n\sigma_{2,\,1} + b_n\sigma_{1,\,1} + a_n}{c_{n-2}\sigma_{2,\,1} + b_{n-2}\sigma_{1,\,1} + a_{n-2}};\\
\sigma_{1,\,n+1} &= \frac{c_{n-1}\sigma_{2,\,1} + b_{n-1}\sigma_{1,\,1} + a_{n-1}}{c_{n-2}\sigma_{2,\,1} + b_{n-2}\sigma_{1,\,1} + a_{n-2}}.
\end{aligned}
$$

The equations deduced in theorem I are independent of the choice of (p_n, q_n). In order to extend the continued fraction algorithm it is desirable to have $A_n < B_n < C_n$, and to have the A's, B's and C's each form a positive increasing series of integers. Such an expansion will be called *regular*. The defining equations may be written in the form

$$\sigma_{1,\,n} = p_n + \frac{1}{\sigma_{2,\,n+1}}; \qquad \sigma_{2,\,n} = q_n + \frac{\sigma_{1,\,n+1}}{\sigma_{2,\,n+1}}.$$

Jacobi* chose p_n and q_n as $[\sigma_{1,\,n}]$ and $[\sigma_{2,\,n}]$ respectively, where the symbol $[x]$ denotes the greatest integer in x. We notice that the equation defining $\sigma_{1,\,n}$ is the defining equation for ordinary continued fractions, while that for $\sigma_{2,\,n}$ is not. We note also that the expansion may remain regular under other conditions, which we state as

THEOREM II. *If u_1, v_1, w_1, three positive numbers, such that $u_1 < v_1 < w_1$, be expanded into a ternary continued fraction, the laws of selecting (p_n, q_n) being $p_n = [v_n/u_n]$, $0 < q_n \leqq w_n/u_n$, and $[q_1 \geqq p_1]$, then the expansion is regular as defined above.*

This selection of (p_n, q_n) is such that u_n, v_n, and w_n are always positive, as well as p_n and q_n, and from the recursion formulæ (3) it follows that A_n, B_n, and C_n each form a positive and increasing series of numbers. Since $A_1 = 1$, $B_1 = p_1$, and $C_1 = q_1$, we have

$$A_1 \leqq B_1 \leqq C_1.$$

Also $A_2 = q_2$, $B_2 = p_1 q_2 + 1$, $C_2 = q_1 q_2 + p_2$, and since $p_1 \geqq 1$, $p_2 \geqq 0$, and $q_1 \geqq p_1$, we have

$$A_2 < B_2 \leqq C_2,$$

the equality sign holding if $p_2 = 0$, $q_2 = 1$ and $q_1 = p_1 + 1$, for if $q_1 > p_1$, then $q_1 \geqq p_1 + 1$ and $C_2 \geqq (p_1 + 1)q_2 + p_2 \geqq B_2$. The equality sign also holds if $q_1 = p_1$, for in that case (since $w_1 > u_1$) $v_2 > u_2$ and consequently $p_2 \geqq 1$ and $C_2 \geqq q_1 q_2 + 1 \geqq B_2$. Now

$$\begin{aligned} A_3 &= q_3 A_2 + p_3 A_1 + 0, \\ B_3 &= q_3 B_2 + p_3 B_1 + 0, \\ C_3 &= q_3 C_2 + p_3 C_1 + 1. \end{aligned}$$

Therefore $A_3 < B_3 < C_3$, and in general $A_n < B_n < C_n$ for $n \geqq 3$.

Although the above conditions are sufficient, they are not necessary. One exception is that q_n may be zero, provided that p_n be large enough.

An important case of a regular expansion is one which may be considered as an extension of an ordinary continued fraction, in which the

* Jacobi, C. G. J., *Ges. Werke*, VI (385–426).

partial quotients are palindromic.* In this case, however, we define a skew-palindromic expansion as one in which the q's are palindromic and the p's, omitting the first, are also palindromic.

THEOREM III. *If u_1, v_1, w_1, three rational numbers, relatively prime, be expanded into a regular ternary continued fraction, then the necessary and sufficient condition that the series $u_1, u_2, \cdots u_n$ be the series $A_n, A_{n-1}, \cdots A_1$ (i.e., $A_k = u_{n-k+1}$), where (A_n, B_n, C_n) is the last convergent set, is that after the first partial quotient set, the expansion be skew-palindromic.*

By means of equations (1) we have

$$u_k = w_{k+1} = q_{k+1}u_{k+1} + v_{k+2}$$

and consequently

$$u_k = q_{k+1}u_{k+1} + p_{k+2}u_{k+2} + u_{k+3}. \tag{6}$$

The conditions imposed on p_k and q_k may be written

$$p_k = p_{n-k+3}; \qquad q_k = q_{n-k+2}. \tag{7}$$

Since we have selected u_1, v_1, and w_1 rational and relatively prime, we have $u_n = 1$, $v_n = p_n$, and $w_n = q_n$.

In general

$$A_{n-k+1} = q_{n-k+1}A_{n-k} + p_{n-k+1}A_{n-k-1} + A_{n-k-2}, \tag{8}$$

and in particular $A_0 = 0$; $A_1 = 1 = u_n$; $A_2 = q_2 = w_n = u_{n-1}$. The proof is by induction. Assume $A_k = u_{n-k+1}$ for all values up to and including k. Then

$$A_{k+1} = q_{k+1}A_k + p_{k+1}A_{k-1} + A_{k-2} \tag{8'}$$

and

$$u_{n-k} = q_{n-k+1}u_{n-k+1} + p_{n-k+2}u_{n-k+2} + u_{n-k+3}. \tag{6'}$$

And by comparison, using our assumption and equations (7), we get

$$A_{k+1} = u_{n-(k+1)+1} = u_{n-k}.$$

And since $A_k = u_{n-k+1}$ for $k = 0, 1, 2$, it follows that it is true for all values of k.

Conversely if $A_k = u_{n-k+1}$ for all values of k, then equations (7) are true. For by subtracting equation (6′) from (8′) we have, using this assumed equation,

$$0 = A_k(q_{k+1} - q_{n-k+1}) + (p_{k+1} - p_{n-k+2})A_{k-1}. \tag{9}$$

Since $A_2 = q_2$, $q_n = u_{n-1}$ and we have assumed $A_2 = u_{n-1}$, it follows from (9), by putting $k = n - 1$, that $p_3 = p_n$. Assume $q_k = q_{n-k+2}$ and $p_{k+1} = p_{n-k+2}$ for all values up to k. It follows by comparing equations (6′) and

* A sequence of numbers is said to be palindromic, when it reads the same backwards as forwards.

(8′) that

$$q_{k+1} = q_{n-(k+1)+2}$$

and from (9)

$$p_{(k+1)+1} = p_{n-(k+1)+2}.$$

Hence the theorem is true.

As a consequence of this theorem, we state as a

COROLLARY. *Under the hypothesis of theorem III*

$$B_n = p_1A_n + A_{n-1} \quad \text{and} \quad C_n = q_1A_n + p_2A_{n-1} + A_{n-2}.$$

By using the particular values $w_{n+1} = 1$, $v_{n+1} = 0$ and $u_{n+1} = 0$ in equations (2), we obtain

$$A_n = u_1, \qquad B_n = v_1, \qquad C_n = w_1.$$

And since $v_1 = p_1u_1 + u_2$ and $w_1 = q_1u_1 + p_2u_2 + u_3$, we obtain the equations stated in the corollary.

After these preliminary considerations we come to the following theorem, which is fundamental in the considerations of periodic expansions.

THEOREM IV. *If in a ternary continued fraction, the expansion become periodic after a finite number of terms, then $\sigma_{1,1}$ and $\sigma_{2,1}$ are roots of cubic equations with rational coefficients.*

If we think of the σ's as recurring, obviously the p's and q's recur, and conversely, if we think of the p's and q's as recurring, then the σ's recur also, for it does not matter which period we take to determine these cubic equations.

Consider first the case where the expansion is purely periodic, i.e., where $\sigma_{1,1} = \sigma_{1,n+1}$ and $\sigma_{2,1} = \sigma_{2,n+1}$. Then from theorem I we have

$$\sigma_{1,1} = \frac{B_n\sigma_{2,1} + B_{n-1}\sigma_{1,1} + B_{n-2}}{A_n\sigma_{2,1} + A_{n-1}\sigma_{1,1} + A_{n-2}}; \qquad \sigma_{2,1} = \frac{C_n\sigma_{2,1} + C_{n-1}\sigma_{1,1} + C_{n-2}}{A_n\sigma_{2,1} + A_{n-1}\sigma_{1,1} + A_{n-2}}.$$

Solving the first of these equations for $\sigma_{2,1}$ and substituting in the second, we get

$$\sigma_{2,1} = \frac{-A_{n-1}\sigma_{1,1}^2 + (B_{n-1} - A_{n-2})\sigma_{1,1} + B_{n-2}}{A_n\sigma_{1,1} - B_n}$$

$$= \frac{C_n\left[\dfrac{-A_{n-1}\sigma_{1,1}^2 + (B_{n-1} - A_{n-2})\sigma_{1,1} + B_{n-2}}{A_n\sigma_{1,1} - B_n}\right] + C_{n-1}\sigma_{1,1} + C_{n-2}}{A_n\left[\dfrac{-A_{n-1}\sigma_{1,1}^2 + (B_{n-1} - A_{n-2})\sigma_{1,1} + B_{n-2}}{A_n\sigma_{1,1} - B_n}\right] + A_{n-1}\sigma_{1,1} + A_{n-2}}.$$

And by multiplying up and combining terms we write

$$\sigma_{2,1} = \frac{b_{n-2}\sigma_{1,1}^2 + (a_{n-2} - b_{n-1})\sigma_{1,1} - a_{n-1}}{-c_{n-2}\sigma_{1,1} + c_{n-1}},$$

and equating values of $\sigma_{2,1}$

$$[-A_{n-1}\sigma_{1,1}^2+(B_{n-1}-A_{n-2})\sigma_{1,1}+B_{n-2}][-c_{n-2}\sigma_{1,1}+c_{n-1}]$$
$$=[A_n\sigma_{1,1}-B_n][b_{n-2}\sigma_{1,1}^2+(a_{n-2}-b_{n-1})\sigma_{1,1}-a_{n-1}].$$

Finally

$$E_1\sigma_{1,1}^3+F_1\sigma_{1,1}^2+G_1\sigma_{1,1}+H_1=0, \tag{10}$$

where $E_1=A_nb_{n-2}-A_{n-1}c_{n-2}$.

$$F_1=A_{n-1}c_{n-1}+B_{n-1}c_{n-2}-A_{n-2}c_{n-2}-B_nb_{n-2}+A_na_{n-2}-A_nb_{n-1}.$$
$$G_1=B_{n-2}c_{n-2}-B_{n-1}c_{n-1}+A_{n-2}c_{n-1}-B_na_{n-2}+B_nb_{n-1}-A_na_{n-1}.$$
$$H_1=B_na_{n-1}-B_{n-2}c_{n-1}.$$

In a similar fashion by eliminating $\sigma_{1,1}$, we get the following equation:

$$E_2\sigma_{2,1}^3+F_2\sigma_{2,1}^2+G_2\sigma_{2,1}+H_2=0, \tag{11}$$

where $E_2=-E_1$.

$$F_2=A_nb_n+C_nb_{n-2}-A_{n-2}b_{n-2}-C_{n-1}c_{n-2}+A_{n-1}a_{n-2}-A_{n-1}c_n.$$
$$G_2=C_{n-2}b_{n-2}-C_nb_n+A_{n-2}b_n-C_{n-1}a_{n-2}+C_{n-1}c_n-A_{n-1}a_n.$$
$$H_2=C_{n-1}a_n-C_{n-2}b_n.$$

These equations have previously been obtained by Lehmer,* by another method.

If the expansion is not purely periodic, but say the κth set of σ's is the first to recur, then $\sigma_{1,\kappa}$ and $\sigma_{2,\kappa}$ are roots of cubic equations, and since $\sigma_{1,1}$ and $\sigma_{2,1}$ are linear fractional functions of $\sigma_{1,\kappa}$, $\sigma_{2,\kappa}$, $\sigma_{1,1}$ and $\sigma_{2,1}$ are also roots of cubic equations.

In what follows we shall confine our attention to the case where $\sigma_{1,1}=\theta$ and $\sigma_{2,1}=\theta^2$, θ being the real cube root of D, an integer, but not a perfect cube. It will be convenient to use as well as the complete quotient set (u_n, v_n, w_n) composed of linear functions of θ and θ^2, the rationalized complete quotient set

$$(\bar{u}_n, \bar{v}_n, \bar{w}_n)=(\alpha_n, \alpha_n'+\beta_n'\theta+\gamma_n'\theta^2, \alpha_n''+\beta_n''\theta+\gamma_n''\theta^2),$$

where $\alpha_n=$ Norm of u_n and

$$(\alpha_n'+\beta_n'\theta+\gamma_n'\theta^2)=\frac{v_nN(u_n)}{u_n},$$

and

$$(\alpha_n''+\beta_n''\theta+\gamma_n''\theta^2)=\frac{w_nN(u_n)}{u_n}.$$

The quantities $\bar{u}_n$, $\bar{v}_n$, and $\bar{w}_n$ may have a common integral factor, which is not to be removed as was done by Jacobi.† (See the illustrative example after theorem IX.)

* Lehmer, D. N., *Proc. Nat. Acad. Sciences*, Vol. 4, Dec., 1918 (360–364).
† Jacobi, C. G. J., *Ges. Werke*, VI (385–426).

THEOREM V. *If 1, θ, θ^2 expand into a ternary continued fraction which ultimately becomes periodic, and if m_k be the value of the Pellian cubic for (A_k, B_k, C_k), then the series m_k also becomes periodic.*

Suppose that there are t non-recurring partial and complete quotient sets, after which the expansion becomes periodic, the period containing n terms. Then $u_{t+k} : v_{t+k} : w_{t+k} = u_{t+k+n} : v_{t+k+n} : w_{t+k+n}$ (for all positive k's, since the ratios involved are the roots obtained from cubics computed from identical periods). Since

$$w_{t+k} = u_{t+k-1} \quad \text{and} \quad w_{t+k+n} = u_{t+k+n-1}$$

we can write

$$\frac{u_{t+n+1}}{u_{t+n}} = \frac{u_{t+1}}{u_t}, \qquad \frac{u_{t+n+1}}{u_{t+1}} = \frac{v_{t+n+1}}{v_{t+1}},$$

$$\frac{u_{t+n+2}}{u_{t+n+1}} = \frac{u_{t+2}}{u_{t+1}}, \qquad \frac{u_{t+n+2}}{u_{t+2}} = \frac{v_{t+n+2}}{v_{t+2}},$$

etc.

It follows that

$$u_{t+n+k} = \lambda u_{t+k} \quad (k = 0, 1, \cdots k) \qquad \text{where } \lambda = \frac{u_{t+n}}{u_t}, \text{ and} \tag{12}$$

$$v_{t+n+k} = \lambda v_{t+k} \quad (k = 1, 2, \cdots k).$$

By referring to theorem I, we have

$$A_{t+n}w_{t+n+1} + A_{t+n-1}v_{t+n+1} + A_{t+n-2}u_{t+n+1} = 1,$$

and consequently

$$\lambda(A_{t+n}w_{t+1} + A_{t+n-1}v_{t+1} + A_{t+n-2}u_{t+1}) = 1.$$

Therefore λ is a unit in the domain $k(\theta)$, i.e., $N(\lambda) = 1$. Furthermore, beginning with $k = 0$, $N(u_{t+k})$ forms a periodic series of numbers. Also, if we consider the rationalized complete quotient sets, we have that $(\alpha_k,\ \alpha'_k + \beta'_k\theta + \gamma'_k\theta^2,\ \alpha''_k + \beta''_k\theta + \gamma''_k\theta^2)$ is identical with $(\alpha_{k+n},\ \alpha'_{k+n} + \beta'_{k+n}\theta + \gamma'_{k+n}\theta^2,\ \alpha''_{k+n} + \beta''_{k+n}\theta + \gamma''_{k+n}\theta^2)$ for all values of $k > t$. To complete the proof we shall show that

$$m_{t+k-1} = \frac{\beta'^3_{t+k} - D\gamma'^3_{t+k}}{(\beta'_{t+k}\gamma''_{t+k} - \beta''_{t+k}\gamma'_{t+k})^2}\alpha_{t+k}, \tag{13}$$

and since the values of α_{t+k} and its constant multiplier are unchanged by increasing the subscript by n, it results that the series m_k is also periodic if $k \geqq t$.

By theorem I we have

$$\theta = \frac{B_k(\alpha''_{k+1}+\beta''_{k+1}\theta+\gamma''_{k+1}\theta^2)+B_{k-1}(\alpha'_{k+1}+\beta'_{k+1}\theta+\gamma'_{k+1}\theta^2)+B_{k-2}\alpha_{k+1}}{A_k(\alpha''_{k+1}+\beta''_{k+1}\theta+\gamma''_{n+1}\theta^2)+A_{k-1}(\alpha'_{k+1}+\beta'_{k+1}\theta+\gamma'_{k+1}\theta^2)+A_{k-2}\alpha_{k+1}},$$

$$\theta^2 = \frac{C_k(\alpha''_{k+1}+\beta''_{k+1}\theta+\gamma''_{k+1}\theta^2)+C_{k-1}(\alpha'_{k+1}+\beta'_{k+1}\theta+\gamma'_{k+1}\theta^2)+C_{k-2}\alpha_{k+1}}{A_k(\alpha''_{k+1}+\beta''_{k+1}\theta+\gamma''_{k+1}\theta^2)+A_{k-1}(\alpha'_{k+1}+\beta'_{k+1}\theta+\gamma'_{k+1}\theta^2)+A_{k-2}\alpha_{k+1}}.$$

Multiplying up and equating coefficients of like irrationalities, we get, dropping the subscripts on α, β, etc.,

$$
\begin{aligned}
&C_k\alpha'' + C_{k-1}\alpha' + C_{k-2}\alpha = DA_k\beta'' + DA_{k-1}\beta' = DB_k\gamma'' + DB_{k-1}\gamma',\\
(14)\quad &B_k\alpha'' + B_{k-1}\alpha' + B_{k-2}\alpha = C_k\beta'' + C_{k-1}\beta' = DA_k\gamma'' + DA_{k-1}\gamma',\\
&A_k\alpha'' + A_{k-1}\alpha' + A_{k-2}\alpha = B_k\beta'' + B_{k-1}\beta' = C_k\gamma'' + C_{k-1}\gamma'.
\end{aligned}
$$

If we eliminate α'' and α' from the first set of the above equations, we get

$$
(15)\quad \begin{vmatrix} C_k & C_{k-1} & DA_k\beta'' + DA_{k-1}\beta' - C_{k-2}\alpha \\ B_k & B_{k-1} & C_k\beta'' + C_{k-1}\beta' - B_{k-2}\alpha \\ A_k & A_{k-1} & B_k\beta'' + B_{k-1}\beta' - A_{k-2}\alpha \end{vmatrix} = 0 = \beta''\Delta_1 + \beta'\Delta_2 - \alpha,
$$

where

$$
\Delta_1 = \begin{vmatrix} C_k & C_{k-1} & DA_k \\ B_k & B_{k-1} & C_k \\ A_k & A_{k-1} & B_k \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} C_k & C_{k-1} & DA_{k-1} \\ B_k & B_{k-1} & C_{k-1} \\ A_k & A_{k-1} & B_{k-1} \end{vmatrix},
$$

and

$$
\begin{vmatrix} C_k & C_{k-1} & C_{k-2} \\ B_k & B_{k-1} & B_{k-2} \\ A_k & A_{k-1} & A_{k-2} \end{vmatrix} = 1.
$$

By use of the values of C_{k-1}, B_{k-1}, A_{k-1} obtained from the second group of equations (14), we can reduce Δ_1 and Δ_2 to functions of m_k and the coefficients α, β', $\cdots$ etc.

$$
\begin{aligned}
\Delta_1 &= -\frac{1}{\beta'}\begin{vmatrix} C_k & DA_k & DA_k\gamma'' + DA_{k-1}\gamma' - \beta''C_k \\ B_k & C_k & C_k\gamma'' + C_{k-1}\gamma' - \beta''B_k \\ A_k & B_k & B_k\gamma'' + B_{k-1}\gamma' - \beta''A_k \end{vmatrix}\\
&= -\frac{\gamma'}{\beta'}\begin{vmatrix} C_k & DA_k & DA_{k-1} \\ B_k & C_k & C_{k-1} \\ A_k & B_k & B_{k-1} \end{vmatrix}\\
&= -\frac{\gamma'}{\beta'^2}\begin{vmatrix} C_k & DA_k & DB_k\gamma'' + DB_{k-1}\gamma' - DA_k\beta'' \\ B_k & C_k & DA_k\gamma'' + DA_{k-1}\gamma' - C_k\beta'' \\ A_k & B_k & C_k\gamma'' + C_{k-1}\gamma' - B_k\beta'' \end{vmatrix}\\
&= -\frac{\gamma'\gamma''}{\beta'^2}m_k + \frac{\gamma'^2}{\beta'^2}\begin{vmatrix} C_k & DA_k & DB_{k-1} \\ B_k & C_k & DA_{k-1} \\ A_k & B_k & C_{k-1} \end{vmatrix}, \quad \left[\text{since } m_k = \begin{vmatrix} C_k & DA_k & DB_k \\ B_k & C_k & DA_k \\ A_k & B_k & C_k \end{vmatrix}\right]\\
&= -\frac{\gamma'\gamma''}{\beta'^2}m_k - \frac{\gamma'^2}{\beta'^3}\begin{vmatrix} C_k & DA_k & DC_k\gamma'' + DC_{k-1}\gamma' - DB_k\beta'' \\ B_k & C_k & DB_k\gamma'' + DB_{k-1}\gamma' - DA_k\beta'' \\ A_k & B_k & DA_k\gamma'' + DA_{k-1}\gamma' - C_k\beta'' \end{vmatrix}\\
&= -\frac{\gamma'\gamma''}{\beta'^2}m_k + \frac{\gamma'^3}{\beta'^3}D\Delta_1 + \frac{\gamma'^2\beta''}{\beta'^3}m_k.
\end{aligned}
$$

Solving for Δ_1,

$$
(13a)\qquad \Delta_1 = \frac{\gamma'^2\beta'' - \gamma'\gamma''\beta'}{\beta'^3 - D\gamma'^3}m_k.
$$

and

$$u_{n+1} = a_{n-2} + b_{n-2}\theta + c_{n-2}\theta^2.$$

By replacing u_{n+1} by its value, multiplying out and collecting coefficients, we get

$$\begin{aligned} u_{n+1}(C_n + B_n\theta + A_n\theta^2) &= P_1 + Q_1\theta + R_1\theta^2 \text{ say} \\ &= (C_n a_{n-2} + DB_n c_{n-2} + DA_n b_{n-2}) \\ &+ (C_n b_{n-2} + \quad B_n a_{n-2} + DA_n c_{n-2})\theta \\ &+ (C_n c_{n-2} + \quad B_n b_{n-2} + \quad A_n a_{n-2})\theta^2. \end{aligned}$$

Writing the coefficients as determinants and reducing by means of (14′) as was done in theorem V, we have

$$R_1 = 0.$$

$$Q_1 = \begin{vmatrix} C_n & C_{n-1} & DA_n \\ B_n & B_{n-1} & C_n \\ A_n & A_{n-1} & B_n \end{vmatrix} = \Delta_1 = 0.$$

$$P_1 = \begin{vmatrix} C_n & C_{n-1} & DB_n \\ B_n & B_{n-1} & DA_n \\ A_n & A_{n-1} & C_n \end{vmatrix} = \Delta_3 = 1.$$

Hence equation (20) is true for $k = 1$, since we have just shown

$$1 = u_1 = u_{n+1}(C_n + B_n\theta + A_n\theta^2).$$

Similarly

$$\begin{aligned} u_{n+1}(C_{n-1} + B_{n-1}\theta + A_{n-1}\theta^2) &= P_2 + Q_2\theta + R_2\theta^2 \\ &= (C_{n-1}a_{n-2} + DB_{n-1}c_{n-2} + DA_{n-1}b_{n-2}) \\ &+ (C_{n-1}b_{n-2} + \quad B_{n-1}a_{n-2} + DA_{n-1}c_{n-2})\theta \\ &+ (C_{n-1}c_{n-2} + \quad B_{n-1}b_{n-2} + \quad A_{n-1}a_{n-2})\theta^2. \end{aligned}$$

$R_2 = 0.$

$$Q_2 = \begin{vmatrix} C_n & C_{n-1} & DA_{n-1} \\ B_n & B_{n-1} & C_{n-1} \\ A_n & A_{n-1} & B_{n-1} \end{vmatrix} = \Delta_2 = 1.$$

$$P_2 = \begin{vmatrix} C_n & C_{n-1} & DB_{n-1} \\ B_n & B_{n-1} & DA_{n-1} \\ A_n & A_{n-1} & C_{n-1} \end{vmatrix} = \begin{vmatrix} C_n & C_{n-1} & DC_n - DB_n\beta'' \\ B_n & B_{n-1} & DB_n - DA_n\beta'' \\ A_n & A_{n-1} & DA_n - \quad C_n\beta'' \end{vmatrix} = -\beta''\Delta_3 = -\beta''.$$

Hence $u_{n+1}(C_{n-1} + B_{n-1}\theta + A_{n-1}\theta^2) = -\beta'' + \theta$, while $u_2 = -p_1 + \theta$. Since $\beta'' = p_{n+2} - p_2$, it follows that in order that equation (20) be true $p_{n+2} = p_1 + p_2$, which we stated as a condition for a normal expansion.

And again $u_3 = (p_1p_2 - q_1) - p_2\theta + \theta^2$, while

$$\begin{aligned} u_{n+1}(C_{n-2} + B_{n-2}\theta + A_{n-2}\theta^2) &= P_3 + Q_3\theta + R_3\theta^2 \\ &= (C_{n-2}a_{n-2} + DB_{n-2}c_{n-2} + DA_{n-2}b_{n-2}) \\ &+ (C_{n-2}b_{n-2} + \quad B_{n-2}a_{n-2} + DA_{n-2}c_{n-2})\theta \\ &+ (C_{n-2}c_{n-2} + \quad B_{n-2}b_{n-2} + \quad A_{n-2}a_{n-2})\theta^2. \end{aligned}$$

$R_3 = 1.$

$$Q_3 = \begin{vmatrix} C_n & C_{n-1} & DA_{n-2} \\ B_n & B_{n-1} & C_{n-2} \\ A_n & A_{n-1} & B_{n-2} \end{vmatrix} = \begin{vmatrix} C_n & C_{n-1} & -DA_n\alpha'' - DA_{n-1}\alpha' + DC_n \\ B_n & B_{n-1} & -C_n\alpha'' - C_{n-1}\alpha' + DB_n \\ A_n & A_{n-1} & -B_n\alpha'' - B_{n-1}\alpha' + DA_n \end{vmatrix}$$
$$= -\alpha''\Delta_1 - \alpha'\Delta_2 = -\alpha'.$$

Hence we must have $-\alpha' = -(p_{n+1} - p_1) = -p_2$ or $p_{n+1} = p_1 + p_2$, a condition of our hypothesis.

$$P_3 = \begin{vmatrix} C_n & C_{n-1} & DB_{n-2} \\ B_n & B_{n-1} & DA_{n-2} \\ A_n & A_{n-1} & C_{n-2} \end{vmatrix} = \begin{vmatrix} C_n & C_{n-1} & -DB_n\alpha'' - DB_{n-1}\alpha' + D^2A_n \\ B_n & B_{n-1} & -DA_n\alpha'' - DA_{n-1}\alpha' + DC_n \\ A_n & A_{n-1} & -C_n\alpha'' - C_{n-1}\alpha' + DB_n \end{vmatrix}$$
$$= -\alpha''\Delta_3 - \alpha'P_2 + D\Delta_1 = \alpha'\beta'' - \alpha''.$$

Hence we must have

$$p_1p_2 - q_1 = \alpha'\beta'' - \alpha''$$
$$= (p_{n+1} - p_1)(p_{n+2} - p_2) - (p_1p_2 - q_1 - p_1q_{n+2} + q_{n+1}).$$

Inserting the conditions $p_{n+1} - p_1 = p_2$ and $p_{n+2} - p_2 = p_1$, and solving for q_{n+1}, we have $q_{n+1} = p_1^2 + 2q_1$, a condition of the hypothesis. And hence it follows that equation (20) is true for $k = 3$, and hence for all values of k. Consequently, since $N(u_{n+1}) = 1$, the proposed theorem is true, that is $(C_{n-k+1}, B_{n-k+1}, A_{n-k+1})$ is a solution of $x^3+Dy^3+D^2z^3-3Dxyz=\alpha_k$.

It follows that the series of m's is purely periodic, its period being the reverse of the period of α_k. We may interpret n as the number of terms in any number of periods, for if the expansion is normal, then it is normal whether we consider only one period or any number of periods.

We state as a converse to theorem IX: *If the expansion of* 1, θ, θ^2 *have one non-recurring* q *and two non-recurring* p's, *after which it be periodic, and if* $\alpha_k = m_{n-k+1}$ *for all positive values of* k, *then the expansion is a normal expansion.*

We saw in theorem VIII that, under the conditions of the hypothesis, the series of α's is purely periodic, and that $N(u_{n+1}) = 1$. If $N(u_k) = N(C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2)$, then $u_k = \lambda(C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2)$, where $N(\lambda) = 1$. If we put $k = 1$, this reduces to $u_1 = 1 = \lambda(C_n + B_n\theta + A_n\theta^2)$, from which $\lambda = u_{n+1}$ and

$$u_k = (a_{n-2} + b_{n-2}\theta + c_{n-2}\theta^2)(C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2).$$

But we have seen that this equation can not be true for $k = 2$ and $k = 3$ unless $p_{n+1} = p_{n+2} = p_1 + p_2$ and $q_{n+1} = p_1^2 + 2q_1$. Further, we may write $u_{k+1} = u_{n+1}(C_{n-k} + B_{n-k}\theta + A_{n-k}\theta^2)$ in the form of equation (21), from which it follows that $p_k = p_{n-k+3}$ and $q_{k-1} = q_{n-k+3}$, and the converse is established.

We give as an illustrative example the expansion for $D = 13$, showing the expansion when u_n has not been rationalized, and when it has.

α	u	v	w	$p,\ q$	A	B	C	m
					1	0	0	
					0	1	0	
					0	0	1	
1	1	θ	θ^2	2, 5	1	2	5	8
5	$-2+\theta$	$-5+\theta^2$	1	1, 2	2	5	11	18
12	$-3-\theta-\theta^2$	$5-2\theta$	$-2+\theta$	1, 1	3	7	17	12
18	$8-\theta-\theta^2$	$1+2\theta-\theta^2$	$-3-\theta-\theta^2$	1, 1	6	14	33	5
8	$-7+3\theta$	$-11+2\theta^2$	$8-\theta-\theta^2$	1, 2	17	40	94	1
1	$-4-3\theta+2\theta^2$	$22-7\theta-\theta^2$	$-7+3\theta$	3, 14	259	609	1432	8
5	$34+2\theta-7\theta^2$	$49+45\theta-28\theta^2$	$-4-3\theta+2\theta^2$	3, 2	575	1352	3879	18
12	...	...	...	1, 1		...		12

$\alpha=\bar{u}$	$\bar{v}$	$\bar{w}$		
1	θ	θ^2	2, 5	as above
5	$6+3\theta-\theta^2$	$4+2\theta+\theta^2$	1, 2	
12*	$6+6\theta$	$8+2\theta+2\theta^2$	1, 1	
18*	$12+6\theta$	$3+3\theta+3\theta^2$	1, 1	
8	$7+3\theta-\theta^2$	$2+2\theta+2\theta^2$	1, 2	
1	$1+\theta$	$5+2\theta+\theta^2$	3, 14	
5	$16+3\theta-\theta^2$	$4+2\theta+\theta^2$	3, 2	
12	$6+6\theta$	$8+2\theta+2\theta^2$	1, 1	The period closes with the preceding set.

An attempt has been made to calculate a table of normal expansions. A table of expansions from 1 to 28 appears at the end of this paper. Because of the arbitrary choice of q, several expansions, all normal, have been found in many cases. They have not, however, all been recorded in this table.

The method of finding an expansion is largely experimental, and in most cases it is best to find a solution of the Pellian cubic for $m = 1$ first, and then to expand this solution. The rejection of incorrect choices of q_n is facilitated by theorem III, and the expansion is completed by the use of theorem IX. In the cases where a solution of the Pellian cubic is not available, these theorems help in finding such a solution.

* Note here that $\bar{u}, \bar{v}, \bar{w}$ have a common factor, which is not removed, otherwise theorem IX would no longer hold.

TABLE OF NORMAL EXPANSIONS OF $\sqrt[3]{D}$ (EXCEPT AS INDICATED).

D	
2	1, 1; 2, 3; 3*, 3*;
3	1, 2; 0, 2; 1*, 5; 1, 2*;
4	1, 2; 0, 1; 0*, 1; 0, 1; 1, 5; 1, 1*; or 1, 1; 2, 1; 1*, 1; 3, 3; 3, 1*;
5	1, 2; 1, 1; 1*, 2; 3, 2; 1, 1; 2, 5; 2, 1*;
6	1, 2; 1, 1; 0*, 1; 1, 2; 0, 1; 0, 1; 0, 2; 1, 1; 0, 1; 2, 5; 2, 1*;
7[1]	1, 3; 0, 1; 0*, 1; 2, 6; 1, 1*;
9	2, 4; 4, 12; 6*, 12*;
10	2, 4; 4, 6; 3*, 6; 6, 12; 6, 6*;
11	2, 4; 4, 4; 2*, 4; 6, 12; 6, 4*;
12[1]	2, 5; 0, 2; 1*, 1; 0, 1; 1, 3; 2, 2; 2, 1; 1, 1; 2, 2; 1, 3; 6, 13; 2, 2*;
13	2, 5; 1, 2; 1*, 1; 1, 1; 1, 2; 3, 14; 3, 2*;
14	2, 5; 1, 2; 0*, 1; 0, 2; 3, 14; 3, 2*;
15	2, 6; 0, 2; 0*, 5; 0, 1; 0, 1; 0, 3; 0, 1; 0, 1; 0, 5; 0, 2; 2, 16; 2, 2*;
16	2, 6; 0, 1; 1*, 1; 1, 2; 2, 3; 1, 4; 1, 3; 2, 2; 1, 1; 1, 1; 2, 16; 2, 1*;
17	2, 5; 2, 1; 0*, 1; 0, 1; 0, 2; 5, 2; 0, 1; 0, 1; 0, 1; 4, 14; 4, 1*;
18	2, 6; 1, 1; 1*, 1; 2, 1; 1, 1; 3, 16; 3, 1*;
19	2, 6; 1, 1; 0*, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 3, 16; 3, 1*;
20	2, 6; 1, 1; 0*, 1; 0, 2; 1, 4; 1, 2; 0, 1; 0, 1; 3, 16; 3, 1*;
21	2, 6; 2, 1; 2*, 7; 1, 1; 0, 1; 0, 1; 1, 7; 2, 1; 4, 16; 4, 1*;
22	2, 6; 2, 1; 0*, 2; 1, 1; 0, 1; 1, 1; 1, 1; 0, 1; 1, 2; 0, 1; 4, 16; 4, 1*;
23	2, 6; 2, 1; 0*, 1; 2, 2; 0, 1; 0, 1; 1, 3; 0, 1; 0, 1; 1, 3; 0, 5; 1, 1; 12, 135; 12, 1;
	1, 5; 0, 3; 1, 1; 0, 1; 0, 3; 1, 1; 0. 1; 0, 2; 2, 1; 0, 1; 4, 16; 4, 1*;
24	2, 7; 1, 1; 0*, 1; 3, 3; 0, 1; 0, 1; 0, 3; 3, 1; 0, 1; 3, 18; 3, 1*;
25[1]	2, 7; 1, 1; 0*, 1; 3, 7; 1, 1; 0, 2; 3, 3; 1, 1; 0, 4; 1, 2; 0, 1; 0, 1; 4, 6; 1, 1; 0, 1;
	1, 2; 3, 3; 2, 1; 1, 7; 0, 1; 0, 4; 3, 2; 0, 1; 1, 7; 3, 1; 0, 1; 3, 18; 3, 1*;
26	2, 7; 1, 1; 0*, 1; 3, 18; 3, 1*;
28	3, 9; 6, 27; 9*, 27*;
29	
30	3, 9; 6, 9; 3*, 9; 9, 27; 9, 9*.

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* The first and last partial quotient set of the period are indicated by an asterisk.

[1] The expansion is not normal.

CONCERNING COMPACT KÜRSCHÁK FIELDS.

By Vishnu Dattatreya Gokhale.

CONTENTS.

Abstract.—A *Kürschák field* is a *field with a modulus* ("bewertete Körper), in the sense defined by Kürschák in his memoir, "Ueber Limesbildung und die allgemeine Körpertheorie" (Crelle, vol. 142, 1913). This modulus plays, in the general field, essentially the same role as the absolute value in the fields of classical analysis, viz., real number system, complex number system, etc. Kürschák proves that every Kürschák field determines (in the sense of isomorphism) a definite algebraically closed and perfect Kürschák superfield, the smallest such superfield, being (in the sense of isomorphism) a subfield of every such superfield. This definite superfield is *the (smallest) algebraically closed and perfect extension* of the original Kürschák field.

In the present paper the author sets up the notion *compactness*. This notion is analogous to M. Fréchét's compactness and to the J-compactness in E. H. Moore's General Analysis. It is a generalization of the following property in the point set theory: Every infinite set of points in a bounded domain has at least one condensation point. He then studies the properties of algebraically closed and compact fields, and compactness under the adjunction of algebraic elements. Using Ostrowski's results he proves the theorem that the smallest algebraically closed extension of a compact field is compact if, and only if, it can be obtained by adjoining a single algebraic element. The last part of the paper develops a complete existential theory of the four properties: (1) of characteristic other than zero, (2) algebraic closure, (3) perfection, and (4) compactness. Out of the $2^4 = 16$ possibilities 11 are shown to be existent and the remaining 5 non-existent.

I. INTRODUCTION.

1.1. **Field.**—Following Moore,* H. Weber,† and Steinitz,‡ a field is defined in the following manner: §

D 1.1 $$\mathfrak{F} \equiv (\mathfrak{P}\ ;\ +^{\text{ on } \mathfrak{P}\mathfrak{P} \text{ to } \mathfrak{P}.1.2.5.6};\ \times^{\text{ on } \mathfrak{P}\mathfrak{P} \text{ to } \mathfrak{P}.3.4.5.7})$$

where $\mathfrak{P} \equiv [p]$ is a general class of elements $p(p_1, p_2, \text{etc.})$; $+$, $\times$ are single-valued functions, $+(p_1, p_2)$ is a definite element p denoted by $p_1 + p_2$, etc.; and the properties numbered 1–7 are:

$$\left.\begin{array}{lr} (1) & p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3 \\ (2) & p_1 + p_2 = p_2 + p_1 \\ (3) & (p_1 p_2) p_3 = p_1 (p_2 p_3) \\ (4) & p_1 p_2 = p_2 p_1 \\ (5) & p_1(p_2 + p_3) = p_1 p_2 + p_1 p_3 \end{array}\right\} (p_1, p_2, p_3)$$

Here (p_1, p_2, p_3) indicates that *the equations* (1–5) *hold for every choice of the elements* $p_1\ p_2\ p_3$ *of the class* $\mathfrak{P}$.

(6) $$p_1 \cdot p_2 \cdot) \cdot \exists 1\ p \ni p_1 + p = p_2$$

Here the meaning of the logical signs $\cdot) \cdot$, $\exists$, 1, and $\ni$ will be clear from the following reading of the property:—*For every* p_1 *and* p_2, *there exists uniquely an element* p *such that* $p_1 + p = p_2$.

Hence,

(6_0) $$\exists 1\ z^{\mathfrak{P}} : \ni : p :) : p + z = p \cdot pz = z.$$

Here the notation $z^{\mathfrak{P}}$ reads "*z belongs to the class* $\mathfrak{P}$." This unique element z is called the *zero element* of the field.

(7) $$\exists\ p \neq z : p_1 \neq z \cdot p_2 \cdot) \cdot \exists 1\ p \ni p_1 p = p_2,$$

to be read "*There exsists an element p different from the element z* (of 6_0), *and for every such element* p_1 *and every element* p_2 *there exists uniquely an element p such that* $p_1 p = p_2$."

* E. H. Moore, "A Doubly Infinite System of Simple Groups" (Chicago Congress, 1893, pp. 208–242), p. 210. This paper will be referred to as M.

† H. Weber, "Die allgemeinen Grundlagen der Galoischen Gleichungstheorie" (*Mathematische Annalen*, 43, 1893, pp. 521–549), p. 526; also "Algebra" (II edition, 1898), Vol. I, p. 492.

‡ Steinitz, "Algebraische Theorie der Körper" (*Crelle*, 137, 1910, pp. 167–309), p. 172. This paper will be referred to as S.

§ For postulational definitions of a field see:

L. E. Dickson, "Definitions of a Group and a Field by Independent Postulates" (*Trans. A. M. S.*, Vol. 6, 1905, pp. 198–204), p. 202.

E. V. Huntington, "Note on the Definitions of Abstract Groups and Fields by Sets of Independent Postulates" (*Trans. A. M. S.*, Vol. 6. pp. 181–197), pp. 186, 191. This paper also contains a bibliography.

This property can be easily seen to be equivalent to the following:

(7′) $\exists\ p \neq z : \exists u^{\mathfrak{F}} \ni [p\cdot)\cdot pu = p : p \neq z\cdot)\cdot \exists\ p' \epsilon\, pp' = u]$.

Also, we have

(7₀) $\exists 1\, u^{\mathfrak{F}} \ni p\cdot)\cdot pu = [p : p_1 \neq z\cdot p_2 p_1 = z :) : p_2 = z$.

This element u is called the *unit element* of the field, and the element (which can be easily seen to be unique) p' associated with p in the second part of 7′ is called the *reciprocal* of p.

Hereafter we denote the elements p of the class $\mathfrak{P}$ of a field not by p but by f (f_1, f_2, etc.).

1.11. **Characteristic.**—A field of *characteristic* p, in notation $\mathfrak{F}^p$ is defined in the following manner: *

D 1.2 $\qquad \mathfrak{F}^p : \equiv : \mathfrak{F}\cdot ?\cdot \exists\ n \ni nu = z\cdot p \equiv$ the smallest such n.

By using the properties of the field it can be proved that in this case,

(1) p is a prime.
(2) $f \neq z :) : nf = z \cdot \sim \cdot n =$ a multiple of p.

On the other hand, for every positive integer m and a prime p, there exists† one and only one finite field, the so-called Galois field $[p^m]$, with characteristic p.

Fields without any such characteristic are said to be of characteristic zero, in notation $\mathfrak{F}^0$. Henceforth in the notation $\mathfrak{F}^p$ we shall understand p to be an indefinite prime number. Thus every field is of the type $\mathfrak{F}^0$ or $\mathfrak{F}^p$.

1.2. **Algebraic closure, algebraic extensions.**—An *algebraically closed* field $\mathfrak{F}$, in notation $\mathfrak{F}^A$, is thus defined: ‡

D 1.3 $\quad \mathfrak{F}^A : \equiv : \mathfrak{F} : \ni :$

$$n\cdot(f_0, f_1, \cdots, f_n) :) : \exists\ f \ni f_0 f^n + f_1 f^{n-1} + \cdots + f_n = z.$$

An element j *algebraic with respect to* $\mathfrak{F}$, in notation $j^{\text{alg}\,\mathfrak{F}}$ is defined as: §

D 1.4 $\quad j^{\text{alg}\,\mathfrak{F}} : \equiv : (j, \mathfrak{F}) : \ni : \exists^{:1} n\cdot(f_1, \cdots, f_n) \cdot \ni \cdot$

$$(x^n + f_1 x^{n-1} + \cdots + f_n)^{\text{irreducible in } \mathfrak{F}} \cdot j^n + \cdots + f_n = z.$$

This unique integer n is called the *order* of j. Here x is an indeterminate, and j belongs to a field $\mathfrak{F}'$ containing $\mathfrak{F}$ as a subfield, in notation, $\mathfrak{F}' \supset \mathfrak{F}$.

Steinitz has shown‖ how to extend a field by the adjunction of an

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* S, p. 181. Cf. also: J. König:—Einleitung in die allgemeine Theorie der algebraischen Grössen (Leipzig, 1913), p. 408. König uses the terms "orthoïd" and "pseudoorthoid" for fields of characteristic zero and p respectively.

† M, p. 211.

‡ S, p. 260.

§ S, p. 183.

‖ S. p. 197.

algebraic element j. This extension of a field $\mathfrak{F}$, in notation $\mathfrak{F}(j)$, is *the* extension in the sense that it is the smallest and unique (in the sense of isomorphism). He also shows how to get the extension which reduces completely a given polynomial φ irreducible in $\mathfrak{F}$. Steinitz also proves:*

$$\textit{Thm } 1.1 \qquad \mathfrak{F} :) : \exists\ \mathfrak{F}'^{\supset \mathfrak{F}} : \ni : \mathfrak{F}'^{A} \cdot \mathfrak{F}''^{A \cdot \supset \mathfrak{F}} \cdot) \cdot \mathfrak{F}'' \supset \mathfrak{F}'.$$

This field $\mathfrak{F}'$, unique in the sense of isomorphism, we denote by $\mathfrak{F}_A$: *the* (smallest) *extension of* $\mathfrak{F}$ *having the property* A.

Every field $\mathfrak{F}$ has one and only one *prime subfield* † $\mathfrak{P}_0$; $\mathfrak{P}_0$ is isomorphic with the rational number system or the integral number system taken modulo p, according as $\mathfrak{F}$ is of characteristic 0 or p. By *absolute algebraic field,* in notation $\mathfrak{P}$, we mean the algebraically closed extension of such a prime field $\mathfrak{P}_0$ ‡

A field $\mathfrak{F}'$, *algebraic extension of* $\mathfrak{F}$, in notation $\mathfrak{F}'^{\text{alg}\,\mathfrak{F}}$, is thus defined as: §

$$D\ 1.5 \qquad \mathfrak{F}'^{\text{alg}\,\mathfrak{F}} : \equiv : (\mathfrak{F}', \mathfrak{F}) : \ni : \mathfrak{F}' \supset \mathfrak{F} \cdot f'^{-\mathfrak{F}} \cdot) \cdot f'^{\text{alg}\,\mathfrak{F}}.$$

Note: The negative sign denotes the absence of the property. Thus the last implication should be read: *If f' does not belong to $\mathfrak{F}$, it is algebraic with respect to $\mathfrak{F}$.*

1.3. **Kürschák Field.**—A Kürschák field, in notation $\mathfrak{K}$, is thus defined: ||

$$D\ 1.6 \qquad \mathfrak{K} \equiv (\mathfrak{F} ; \|\cdot\|^{\text{on } \mathfrak{F} \text{ to } \mathfrak{A}^{\text{real.} \geqq 0}} .1.2.3.4),$$

where $\mathfrak{F}$ denotes a field, $\mathfrak{A}^{\text{real.} \geqq 0}$ the class of all real non-negative numbers, and the properties 1–4 of the single-valued function $\| \quad \|$ (which will hereafter be called the *modulus*) are:—

$$\left.\begin{array}{ll}
(1) & f = z \cdot \sim \cdot \| f \| = 0 \\
(2) & \| f_1 f_2 \| = \| f_1 \| \, \| f_2 \| \quad \therefore \quad \| u \| = 1 \\
(3) & \| f_1 + f_2 \| \leqq \| f_1 \| + \| f_2 \| \\
(4) & \exists\ f : \ni : \| f \| \neq 0 \cdot \| f \| \neq 1
\end{array}\right\} (f_1, f_2).$$

Hereafter we designate the elements of $\mathfrak{K}$ not by f but by $k(k_1, k_2$, etc.).

1.4. **Limit, Perfect Extension.**—A limit k of a sequence $k_1, k_2, \cdots$ in notation $\{k_n\}$, is thus defined: ¶

$$D\ 1.7 \qquad \underset{n}{L}\, k_n = k : \equiv : (k, \{k_n\}) \ni \underset{n}{L} \| k - k_n \| = 0.$$

* S, p. 287.

† S, p. 180.

‡ S, p. 199.

§ S, p. 198.

|| J. Kürschák, "Uber Limesbildung und allgemeine Körpertheorie" (*Crelle*, 142, 1913, pp. 211–233), p. 211. This paper will be referred to as K.

¶ K, p. 222.

As an immediate consequence of the definition,

Thm 1.2 $\underset{n}{L}\, k_n = k \cdot \underset{n}{L} k_n = k' :) : k = k' \cdot \| k \| = \| k' \| = \underset{n}{L} \| k_n \|.$

The usual properties of limits, for instance: if a sequence has a limit, every subsequence has the same limit and conversely; the sum of limits equals the limit of the sum; etc., follow in the usual manner.

A *Cauchy sequence* $\{k_n\}$, or $\{k_n\}$ *satisfying the Cauchy condition:* in notation $\{k_n\}^{c.c}$: is:

D 1.8 $\{k_n\}^{c.c} : \equiv : \{k_n\} : \ni : e :) : \exists\, n_e \cdot \ni \cdot n > n_c \,.\,) \,.\, \| k_n - k_{n_e} \| < e.$

Here, as further, e is a real positive number. As an immediate consequence, we have

Thm 1.3 $\exists\, \underset{n}{L} k_n \cdot) \cdot \{k_n\}^{c.c}.$

The converse however is not true; e.g., in the field of rational numbers with the absolute value as modulus, not all Cauchy sequences have rational limits. A field $\mathfrak{R}_0$ or a subclass $\mathfrak{R}_0$ (not necessarily a field) of a field $\mathfrak{R}$ for which the converse holds is called *perfect*: in notation $\mathfrak{R}_0^P$. Thus the definition of perfection is: *

D 1.9 $\mathfrak{R}_0^P : \equiv : \mathfrak{R}_0^{\mathfrak{R}\subset} : \ni : \{k_{0n}\}^{c.c} \cdot) \cdot \exists\, k_0 \ni \underset{n}{L}\, k_{0n} = k_0;$

in this definition (as also further, where necessary) we denote the elements of $\mathfrak{R}_0$ by $k_0(k_{01}, k_{02}$, etc.).

By a method analogous to G. Cantor's in building up the real number system from the rational numbers, Kürschák has shown† how to extend a Kürschák field so as to make it perfect. The smallest such extension (unique in the sense of isomorphism) of a $\mathfrak{R}$ we designate notationally by $\mathfrak{R}_A$: *the* (smallest) *extension of* $\mathfrak{R}$ *having the property P.*

Given a $\mathfrak{R}^P$ and $j^{\text{alg}\,\mathfrak{R}}$, Kürschák defines $\| j \|$ so that $\mathfrak{R}(j)$ is a Kürschák field: ‡ in notation $\mathfrak{R}(j)^K$. Defining in $\mathfrak{R}_A$ the modulus of every element algebraic with respect to $\mathfrak{R}$ in the same manner, we have $\mathfrak{R}_A^K$. Making this perfect, we get $\mathfrak{R}_{AP}^P$, in the sense $(\mathfrak{R}_A)_P$. Kürschák then shows§ that $\mathfrak{R}_{AP}$ is algebraically closed. If we start with any Kürschák field $\mathfrak{R}$, we

* K, p. 228.

† K., p. 228.

‡ K, p. 245. Kürschák defines $\|j\|$ to be $\|f_n\|^{1/n}$, where f_n is the f_n in D 1.4. A. Ostrowski in his "Uber sogenannte perfecte Körper" (*Crelle*, 147, 1917, pp. 191–204), p. 196, and "Uber einige Lösungen der Functionalgleichung $\varphi(xy) = \varphi(x)\varphi(y)$" (*Acta Mathematica*, 41, 1918, p. 271–284), p. 280, has shown that this is the only possible definition of $\|j\|$.

These papers will be referred to as O_2, O_3 respectively.

§ K, p. 251.

must first make it perfect and then follow this process. Thus:

$$\mathfrak{R} \rightarrow \mathfrak{R}_P^P \rightarrow \mathfrak{R}_{PA}^A \rightarrow^* \mathfrak{R}_{PAP}^{A.P.}.$$

Kürschák's final theorem is thus: †

Thm 1.4 $\qquad \mathfrak{R} :) : \mathfrak{R}_{PAP}^{A.P.\supset\mathfrak{R}} \cdot \mathfrak{R}'^{A.P.\supset\mathfrak{R}} \cdot) \cdot \mathfrak{R}' \supset \mathfrak{R}_{PAP}$

This perfect and algebraically closed extension of $\mathfrak{R}$ we shall denote by $\mathfrak{R}_{AP}$ or $\mathfrak{R}_{PA}$.

II. COMPACTNESS AND ALGEBRAIC EXTENSIONS.

2.1. **Compactness.**—In classical point set theory we have the theorem that every infinite set of points in a bounded domain has at least one condensation point. A similar property for a general class for which the limit function exists is defined by Fréchét.‡ Classes having this property are said to be, according to his nomenclature, compact. The following definition of compactness in the case of a Kürschák field is naturally suggested as analogous to this definition of Fréchét and the definition of J-compactness in Moore's theory.§

A subset (not necessarily a field) *of a Kürschák field is said to be compact if every infinite sequence of elements of the set such that the* (smallest) *upper bound of the modulus is finite, contains at least one Cauchy subsequence with its limit element in the set.*

In notation:

$$D\ 2.1 \qquad \mathfrak{R}_0^{cpt} : \equiv : \mathfrak{R}^{\subset\mathfrak{R}} : \ni : \{k_{0n}\} \ni \overline{\underset{n}{B}} \| k_{0n} \| < \infty \cdot) \cdot \exists\ (k_0,\ \{n_m\}^{\text{pr. mon. inc.}}) \ni \underset{n}{L}\, k_{0n_m} = k_0.$$

Here as in D 1.9 $\mathfrak{R}_0$ is a subset not necessarily a field of a Kürschák field $\mathfrak{R}$; and $\{n_m\}$ is a *properly monotonic increasing sequence* of positive integers n. This property denoted notationally in the above definition as pr. mon. inc. is thus defined:

$$D\ 2.11 \qquad \{n_m\}^{\text{pr. mon. inc.}} : \equiv : \{n_m\} : \ni : m_1 > m_2 \cdot) \cdot n_{m_1} > n_{m_2}.$$

2.2. **Compactness and Perfection.**—We now prove that:

Thm 2.1 $\qquad \mathfrak{R}_0^{cpt} \cdot) \cdot \mathfrak{R}_0^P.$

* In his theses, "Uber einige Fragen der allgemeinen Körpertheorie" (*Crelle*, 143, 1913, pp. 255–284), p. 284, A. Ostrowski finds under what conditions this step is necessary. He also shows (p. 260) that the algebraically closed extension of Hensel's p-adic numbers is not perfect. This paper will be referred to as O_1.

† K, p. 251.

‡ M. Fréchét, "Sur quelques points du calcul fonctionnel" (*Rendiconti del Circ. Math. d. Palermo*, 22, 1906), p. 6.

§ E. H. Moore, "Lectures on Matrices in General Analysis" (University of Chicago, 1919–1920).

Proof: Consider a Cauchy sequence $\{k_{0n}\}$. Since it is a Cauchy sequence,

$$e:):\exists n_e \ni n > n_e \cdot) \cdot \| k_{0n} - k_{0n_e} \| < e.$$

Therefore for such an n, $\| k_{0n} \|$ lies between $| \, \| k_{0n_e} \| - e \, |$ and $\| k_{0n_e} \| + e$. Hence $\underset{n}{\overline{\mathrm{B}}} \| k_{0n} \| < \infty$. Therefore by compactness we have a subsequence having a limit in $\mathfrak{K}_0$. Hence the original sequence has the same limit in $\mathfrak{K}_0$. Hence the theorem.

Note: The converse of this theorem, however, is not true. See *Thm* 2.3 below. Compare also theorems 3.3 and 3.4 in the next section.

2.3. **Compactness and Algebraic Closure.**—

Thm 2.2 $$\mathfrak{K}^{A.\,\mathrm{cpt}} \cdot a \geqq 0 :) : \exists k \ni \| k \| = a.$$

Proof: The theorem is obvious when $a = 0$ or 1, the corresponding k being z and u respectively. When a is neither the proof is as follows:

By D 1.6 property 4, there exists an element whose modulus is neither 0 nor 1. Let this element be denoted by k_0 and let its modulus be ξ.

By the theory of the real number system $\exists \{r_n\} \ni \underset{n}{L}\, \xi^{r_n} = a$, where $\{r_n\}$ is a sequence of ordinary rationals r. Hence by $\mathfrak{K}^A$,

$$\exists \{k_0^{r_n}\} \cdot \underset{n}{\overline{B}} \| k^{r_n} \| < \infty.$$

Therefore by $\mathfrak{K}^{\mathrm{cpt}}$,

$$\exists (k, \{n_m\}^{\mathrm{pr.\,mon.\,inc.}}) \ni \underset{m}{L}\, k^{r_{n_m}} = k$$

and $\| k \| = \underset{m}{L} \| k_0^{r_{n_m}} \| = \underset{m}{L}\, \xi^{r_{n_m}} = a.$

Using Moore's results,* Steinitz proves: †

Thm 2.3 $$f^{\mathfrak{P}^p} \neq z :) : \exists m \ni f^{p^m - 1} - u = z.$$

We shall use this theorem to prove:

Thm 2.4 $$\mathfrak{K}^{A.\,\mathrm{cpt}} \cdot) \cdot \mathfrak{K}^0.$$

Proof: We shall give a direct proof of a contrapositive of the theorem, viz.,

$$\mathfrak{K}^{p'A} \cdot) \cdot \mathfrak{K}^{-\mathrm{cpt}}.$$

Every $\mathfrak{K}^{p'A}$ contains $\mathfrak{P}^p$, the absolute algebraic field characteristic p. Consider $\{k_n\}^{\mathrm{distinct.}\,\mathfrak{P}^p}$. By theorem 2.3 we have:

$$n :) : \| k_n \| = 0 \qquad \text{or} \qquad \| k_n \|^q = 1,$$

where $q = p^m - 1$ in theorem 2.3. Hence $\underset{n}{\overline{B}} \| k_n \| = 1 \therefore < \infty$. But

* M, p. 220.

† S, p. 251.

this sequence cannot have a Cauchy subsequence; for, otherwise,

$$e:):\exists\ (n_1,\, n_2)^{\text{distinct}} \ni \|\, k_{n_1} - k_{n_2} \,\| < 2e,$$

viz., n_1, n_2 each greater than n_e in D 1.8. But from theorem 2.3 and the fact that $n_1 \neq n_2 \cdot) \cdot \|\, k_{n_1} \,\| \neq \|\, k_{n_2} \,\|$, we have

$$n_1 \neq n_2 \cdot) \cdot \|\, k_{n_1} - k_{n_2} \,\| = 1.$$

Taking $e < 1/2$ we get a contradiction. Thus this particular sequence has no Cauchy subsequence. Hence the theorem.

We have further:

Thm 2.5 $\qquad \mathfrak{R}^{\text{cpt}} \cdot) \cdot \exists\ n \ni \|\, nu \,\| \neq 1.$

Proof: Here also we prove the contrapositive, viz.,

$$\mathfrak{R} \ni n \cdot) \cdot \|\, nu \,\| = 1 :) : \mathfrak{R}^{-\text{cpt}}.$$

Since $\|\, nu \,\| = 1\ (n)$, we have the characteristic zero, and $\|\, ru \,\| = 1\ (r)$ where r is any ordinary rational different from zero. Consider $\{r_n\}^{\text{distinct}}$. We have $\underset{n}{\overline{B}}\, \|\, r_n u \,\| = 1 \therefore < \infty$.

Now the sequence $\{r_n u\}$ cannot have any Cauchy subsequence; for otherwise,

$$e:):\exists\ (n_1,\, n_2)^{\text{distinct}} \ni \|\, r_{n_1}u - r_{n_2}u \,\| < 2e,$$

but since $n_1 \neq n_2 \cdot) \cdot r_{n_1}u - r_{n_2} = (r_{n_1} - r_{n_2})u \neq z \cdot) \cdot \|\, r_{n_1}u - r_{n_2}u \,\| = 1$, taking $e < 1/2$ we get a contradiction. Hence the theorem.

We have also:

Thm 2.6 $\qquad \mathfrak{R}^{\text{cpt}} \cdot) \cdot$ (the first $n \ni \|\, nu \,\| \neq 1)^{\text{prime}}$.

Proof: For otherwise if $n = n_1 n_2$, where $n_1 \leqq n_2 < n$, $nu = n_1 n_2 u = (n_1 u)(n_2 u)$. Hence $\|\, nu \,\| = 1$ which contradicts the hypotheses.

2.4. **Hensel-Kürschák Fields, Ostrowski's Results.**—A Kürschák field where the property 3 of the modulus in D 1.6 is replaced by the stronger property:

(3′) $\qquad \|\, f_1 + f_2 \,\| \leqq$ greater of $(\|\, f_1 \,\|, \|\, f_2 \,\|) \quad (f_1, f_2)$

is called a Hensel-Kürschák field: in notation $\mathfrak{S}$. Thus:

D 2.2 $\qquad \mathfrak{S} \equiv (\mathfrak{F};\ \|\quad \|_{\text{on}\,\mathfrak{F}\,\text{to}\,\mathfrak{R}\,\text{real.}\geqq 0.1.2.3'.4}),$

where the properties 1, 2, 4 are those in D 1.6, and 3′ is defined above. Hereafter we denote the elements of $\mathfrak{S}$ by $h(h_1, h_2,$ etc.); the class of all Hensel-Kürschák fields will be denoted by H, and the superscript H will denote the property of belonging to this class. These notations will be used only when it is necessary to emphasize the fact that the Kürschák field under consideration is a Hensel-Kürschak field.

The modulus of a Hensel-Kürschák field, which, when necessary, we shall call Hensel-Kürschák modulus, is what Ostrowski* calls a non-Archimedian modulus: in notation: $\|\ \ \|^{\text{non-arch}}$. He also proves:

Thm† 2.7 $\quad \|\ \ \|^{\text{non-arch}} : \sim : n \cdot) \cdot \| nu \| \leqq 1.$

Thm‡ 2.8 $\quad \| h_1 \| > \| h_2 \| :) : \| h_1 + h_2 \| = \| h_1 \|.$

A modulus which does not obey this stronger property 3′ is called Archimedian: in notation: $\|\ \ \|^{\text{arch}}$.

A. Ostrowski has investigated real-valued solutions of the functional equations:

$$\left.\begin{aligned} \varphi(xy) &= \varphi(x)\varphi(y) \\ \varphi(x+y) &\leqq \varphi(x) + \varphi(y) \end{aligned}\right\} (x,\ y)^{\text{rational}}.$$

His conclusions, modified for the Kürschák modulus by the fact that the modulus is non-negative, are:

Thm§ 2.9 $\quad \mathfrak{R}^0 :) : \quad (1)\ r \neq z \cdot) \cdot \| r \| = 1;\ \| z \| = 0$

$$\text{or } (2)\ r \cdot) \cdot \| r \| = | r |^{\rho} \qquad 0 < \rho \leqq 1$$
$$\text{or } (3)\ r \cdot) \cdot \| r \| = c^{0(p,\, r)} \qquad 0 < c < 1,\ p^{\text{prime}}.$$

Here $0(p,\ r)$ is *the order of r with respect to p*, defined, following Hensel ‖ as:

D 2.3 $\quad \left[r = \dfrac{u}{v} p^{\rho} \ni (u,\ v)^{\text{prime to } p} \right] :) : 0(p,\ r) \equiv \rho.$

In the above theorem, the modulus is Archimedian in case 2, non-Archimedian in the other two. For $n = 2$ in theorem 2.6, there does exist a field with Archimedian modulus, viz., $\mathfrak{A}^{\text{complex}}$, with the modulus as defined in the second part of theorem 2.9. If we denote this field by $\mathfrak{A}_{\rho}$, and define equivalence: in notation $\sim$: as isomorphism between Kürschák fields which is preserved under the limit process, we have:

Thm 2.10 $\quad \rho_1 \cdot \rho_2 :) : \mathfrak{A}_{\rho_1} \sim \mathfrak{A}_{\rho_2} \quad \begin{pmatrix} 0 < \rho_1 \leqq 1 \\ 0 < \rho_2 \leqq 1 \end{pmatrix}.$

Ostrowski proves:¶

Thm 2.11 $\quad \mathfrak{R}^{-H} \cdot) \cdot \exists\, \rho \ni \mathfrak{R} \subset \mathfrak{A}_{\rho}$

and finally,**

Thm 2.12 $\quad \mathfrak{R}^{-H \cdot P.A.} \cdot) \cdot \mathfrak{R} \sim \mathfrak{A}^{\text{complex}}.$

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* O_3, p. 273.

† O_3, p. 273.

‡ O_3, p. 274.

§ O_3, p. 276.

‖ K. Hensel, "Theorie der algebraischen Zahlen" (Leipzig, 1908, Vol. I).

¶ O_3, p. 281.

** O_3, p. 282.

2.5. **Deductions from Ostrowski's Results.**—From theorem 2.7 the Archimedian or non-Archimedian character of the modulus depends upon the moduli of the elements of the prime subfield. Hence we have:

Thm 2.13 $$\mathfrak{R}' \supset \mathfrak{R} :) : \mathfrak{R}^{H(-H)} \cdot \sim \cdot \mathfrak{R}'^{H(-H)}.$$

Here the two parentheses go together.

From theorem 2.12 we have

Thm 2.14 $$\mathfrak{R}^{-H.P.A.} \cdot) \cdot \mathfrak{R}^{\text{cpt}}.$$

We shall now prove an important theorem to be used in the sequel: viz., *compactness is not extensionally attainable.** To be more precise,

Thm 2.15 $$\mathfrak{R}^{-\text{cpt}.P} \cdot) \cdot - \exists\, \mathfrak{R}'^{\supset \mathfrak{R},\,\text{cpt}}.$$

Proof: Since the field is perfect every Cauchy sequence has a limit element in the field. Want of compactness, therefore, is due to the fact that there exists an infinite sequence which has no Cauchy subsequence. Now every extension preserves the moduli; this fact will therefore persist in every extension. Hence no extension will make the field compact.

We show in section 4 theorem 4.10_0 that $\mathfrak{S}^{-\text{cpt}}_{(PA)}$, where $\mathfrak{S}$ is the Hensel field of p-adic numbers. Since in this case compactness is not extensionally attainable (in virtue of theorem 2.15), we have, using theorems 2.9 and 2.4,

Thm 2.16 $$\mathfrak{R}^{A.\,\text{cpt}} : \sim : \mathfrak{R} \sim \mathfrak{A}^{\text{complex}}.$$

2.6. **Compactness and Adjunction of Algebraic Elements.**—An *algebraic element of the first kind:* in notation alg_1: is thus defined: †

D 2.4 $$j^{\text{alg}_1} : \mathfrak{R} \equiv : (j, \mathfrak{R}) : \ni : j^{\text{alg}\,\mathfrak{R}} : \varphi^{\text{irreducible in }\mathfrak{R}.} \cdot \varphi(j) = z \cdot) \cdot \exists \psi \ni \psi(j) \neq z \cdot \varphi = (. - j)\psi.$$

Note: Here the dot in the last brackets stands for the argument of the function $[x - j \mid x]$: read "$x - j$ as x varies."

An algebraic element, which is not of the first kind is said to be *of the second kind:* in notation: alg_2. Similar definitions hold for algebraic extensions of the first and second kind.

Ostrowski proves: ‡

Thm 2.17 $$\mathfrak{R}^{P} \cdot j^{\text{alg}_1\,\mathfrak{R}} \cdot) \cdot \mathfrak{R}(j)^{P}.$$

We prove the analogous theorem:

Thm 2.18 $$\mathfrak{R}^{\text{cpt}} \cdot j^{\text{alg}_1\,\mathfrak{R}} \cdot) \cdot \mathfrak{R}(j)^{\text{cpt}}.$$

* *A property is said to be extensionally attainable when there exists an extension which has that property.* Compare E. H. Moore, "Introduction to a Form of General Analysis," Yale University Press, 1910, pp. 53, 54.

† S, p. 231.

‡ O_1, p. 275.

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Proof: Let m be the order of j and μ an upper bound of the modulus of an infinite sequence $\{k_{n,1}j^{m-1} + \cdots + k_{n,m} \mid n\}$ of elements of $\mathfrak{R}(j)$. We have to show that this sequence has at least one Cauchy subsequence with a limit element in $\mathfrak{R}(j)$.

Let us denote this sequence by $\{i_{0,n}\}$ and let the m conjugates of $i_{0,n}$ be denoted by $i_{1,n}, i_{2,n}, \cdots, i_{m,n}$; $i_{1,n} \equiv i_{0,n}$ (n). Let the m conjugates of j be denoted by $j_1, j_2, \cdots, j_m$; $j \equiv j_1$.

In the normal extension* of $\mathfrak{R}$ which contains $\mathfrak{R}(j)$ we have for every n the m linear equations:

$$\begin{array}{c} k_{n,1}j_1^{m-1} + \cdots + k_{n,m} = i_{1,n} \\ k_{n,1}j_2^{m-1} + \cdots + k_{n,m} = i_{2,n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ k_{n,1}j_m^{m-1} + \cdots + k_{n,m} = i_{m,n}. \end{array}$$

Since j is of the first kind, the determinant of coefficients involving j and its conjugates is not the zero element. Hence we can solve for the k's and get $k_{n,r}$ (r, n) as linear functions of the i's with coefficients independent of n. Also $\| i_{r,n} \| = \| i_{0,n} \|$ (n, r). Hence $\overline{B}_{n} \| k_{n,r} \| < \infty$ (r). Consider now the sequence $\{k_{n,1}\}$; by $\mathfrak{R}^{\text{cpt}}$ and the result proved just now, this sequence has at least one Cauchy subsequence with a limit element in $\mathfrak{R}$. Take one such Cauchy subsequence and the corresponding subsequence of $\{i_{0,n}\}$, say $\{i'_n\}$. Denote the coefficients of the different powers of j in this sequence by prime letters. As above we can prove that $\{k'_{n,2}\}$ has a Cauchy subsequence. Take the corresponding subsequence of $\{i'_n\}$ which is itself a subsequence of the original sequence. Continuing this process, after a finite number of steps, viz., m steps, we get a subsequence of the original sequence such that the coefficient sequences are all Cauchy sequences. Let this final sequence be denoted by $\{i_n\}$. Then if

$$i_n \equiv k_{n,1}j^{m-1} + \cdots + k_{n,m} \quad (n),$$

we have:

$$n_1 \cdot n_2 :) : \| i_{n_1} - i_{n_2} \| \leqq gr(\| k_{n_1,1} - k_{n_2,1} \|, \cdots \| k_{n_1,m} - k_{n_2,m} \|) \\ gr(\| j \|^{m-1}, \cdots, \| j \|, 1),$$

where gr denotes "the greater of." Hence this subsequence is a Cauchy sequence. Using theorems 2.1 and 2.17 we see that the limit element is in $\mathfrak{R}(j)$. Hence the theorem.

An algebraic extension is said to be *finite* with respect to the original field: in notation fin : when it can be obtained by adjoining a finite number

* S., p. 207.

of algebraic elements to the original field: *

D 2.5 $\quad \mathfrak{R}'^{\text{fin } \mathfrak{R}} : \equiv : (\mathfrak{R}', \mathfrak{R}) : \ni : \exists n \cdot (j_1, j_2, \cdots, j_n)^{\text{alg } \mathfrak{R}} \ni \mathfrak{R}' = \mathfrak{R}(j_1, \cdots, j_n).$

Steinitz proves: †

Thm 2.19 $\quad \mathfrak{R} \cdot (j_1, \cdots, j_n)^{\text{alg}_1 \mathfrak{R}} :) : \exists j^{\text{alg}_1 \mathfrak{R}} \ni \mathfrak{R}(j) = \mathfrak{R}(j_1, \cdots, j_n).$

Ostrowski proves: ‡

Thm 2.20 $\qquad \mathfrak{R}^P :) : \mathfrak{R}_A^{\text{fin } \mathfrak{R}} \cdot) \cdot \mathfrak{R}_A^{\text{alg}_1 \mathfrak{R}}.$

Using theorem 2.16, he then proves§ a theorem, which, in view of theorem 2.19, we can state as:

Thm 2.21 $\qquad \mathfrak{R}^P :) : \mathfrak{R}_A^P \cdot \sim \cdot \exists j^{\text{alg } \mathfrak{R}} \ni \mathfrak{R}(j) = \mathfrak{R}_A.$

We prove the analogous theorem:

Thm 2.22 $\qquad \mathfrak{R}^{\text{cpt}} :) : \mathfrak{R}_A^{\text{cpt}} \cdot \sim \cdot \exists j^{\text{alg } \mathfrak{R}} \ni \mathfrak{R}(j) = \mathfrak{R}_A.$

Proof: Since compactness implies perfection, from 2.21 we have

$$\mathfrak{R}_A^{\text{cpt}} \cdot) \cdot \exists j^{\text{alg } \mathfrak{R}} \ni \mathfrak{R}(j) = \mathfrak{R}_A.$$

The other part of the theorem follows from theorems 2.20 and 2.18.

III. THE HENSEL-KÜRSCHÁK FIELD $\mathfrak{X}_{\mathfrak{z}}$.

3.1. **The Field** $\mathfrak{X}_{\mathfrak{z}}$.—Given a field $\mathfrak{F}$ we build a system: ‖

D 3.1 $\qquad \mathfrak{X}_{\mathfrak{z}} \equiv [\text{all } \varphi^{\text{on } \mathfrak{J} \text{ to } \mathfrak{F}} : \ni : \exists i_\varphi \ni i < i_\varphi \cdot) \cdot \varphi(i) = z].$

Here $\mathfrak{J} \equiv [i]$ is the class of all integers i.

Addition of elements of the system is the usual addition of functions:

D 3.2 $\qquad \varphi_1 + \varphi_2 \equiv (\varphi_1(i) + \varphi_2(i) \mid i) \qquad (\varphi_1, \varphi_2).$

Multiplication is thus defined: ¶

D 3.3 $\qquad \varphi_1\varphi_2 \equiv (\sum_{i_1, i_2 \mid i_1 + i_2 = i} \varphi_1(i_1)\varphi_2(i_2) \mid i) \qquad (\varphi_1, \varphi_2).$

* S, p. 199.

† S, p. 220.

‡ O_1, p. 281.

§ O_1, p. 284.

‖ In this definition it is obvious that when φ is not the zero function the i_φ's in the definition have a maximum. We denote this maximum by $\nu(\varphi)$ or ν_φ or more simply by ν when it is obvious to which element it belongs. Also in this case $\varphi(\nu) \neq z$. In case φ is the zero function such a maximum does not exist; we denote this symbolically by regarding ν as ∞.

¶ It is to be noticed that the summation in this definition is finitely non-zero, and hence no questions like convergence, etc., are involved; for by D 3.1 both i_1 and i_2 have finite lower bounds for which the functional values of φ_1, φ_2 respectively are non-zero, and since $i_1 + i_2 = i$ the upper bounds too will be finite and for values greater than these the products of the functional values will be zero.

We shall hereafter designate the system in D 3.1 with addition and multiplication as defined in 3.2 and 3.3 by $\mathfrak{X}_{\mathfrak{F}}$. We also introduce a notation for a special set of elements of $\mathfrak{X}_{\mathfrak{F}}$.

D 3.4 $\qquad \delta_i \equiv$ (u at i and z elsewhere) $\qquad (i)$.

We now prove the theorem that this system is a field:

Thm 3.1 $\qquad \mathfrak{F} \cdot) \cdot \mathfrak{X}_{\mathfrak{F}}^{F}$.

Proof: The properties 1–6 and hence 6_0 in D 1.1 are easily seen to be satisfied from the fact that $\mathfrak{F}$ is a field; the zero element is in this case the zero function $\varphi(i) = z$ (i). As for the property 7 we shall prove its equivalent 7′. The first part of this property is obvious; also the element δ_0 satisfies the first condition for the unit element, and in fact also the second condition, that is,

$$\varphi \neq z^{\mathfrak{X}_{\mathfrak{F}}} \cdot) \cdot \exists \phi' \ni \varphi\varphi' = \delta_0.$$

Consider the ν belonging to this φ (footnote 3.1); then an effective function φ' is the function φ' with $\varphi'(i) \equiv z$ for $i < -\nu$, $\varphi'(-\nu) \equiv u/\varphi(\nu)$, and $\varphi'(i)$ for $i = -\nu + s$ $(s > 0)$ as the elements of $\mathfrak{F}$ obtained by the sequential solutions for $s = 1, 2, \cdots$, of the equations:

$$\sum_{i_1, i_2 \mid i_1 - i_2 = s} \varphi(\nu + i_1)\varphi'(-\nu + i_2) = z \qquad (s > 0).$$

Note 1: If we denote symbolically δ_i by x^i, the elements of the field $\mathfrak{X}_{\mathfrak{F}}$ can be symbolically expressed as infinite series $\sum_i f_i x^i$, where $f_i \equiv \varphi(i)$ (i). Addition and multiplication can then be regarded as the corresponding formal operations on the infinite series. We shall, therefore, hereafter take these infinite series as the elements of $\mathfrak{X}_{\mathfrak{F}}$ and operate with them since this procedure is formally more convenient. We shall sometimes denote the range of summation of the index i as from ν to ∞, ν being the integer defined in footnote 3.1.

Note 2: We now identify (in the sense of isomorphism) this field with the field $\mathfrak{F}(x)$ of Steinitz, where x is transcendental* with respect to $\mathfrak{F}$. Putting the various powers of x in one field into correspondence with the same powers of x in the other, and the elements of $\mathfrak{F}$ with the same elements of $\mathfrak{F}$ in the other, we see that $\mathfrak{X}_{\mathfrak{F}} \supset \mathfrak{F}[x]$, where $\mathfrak{F}[x]$ is the integral domain obtained by adjoining the transcendental x. Since the field $\mathfrak{F}(x)$ is obtained from this integral domain by the process of building up of quotients,† we see that $\mathfrak{X}_{\mathfrak{F}} \supset \mathfrak{F}(x)$. On the other hand it is obvious that $\mathfrak{X}_{\mathfrak{F}} \subset \mathfrak{F}(x)$. Hence the result required.

* S, p. 124.

† S, p. 178.

We shall therefore now speak of this process of forming $\mathfrak{X}_\mathfrak{F}$ from $\mathfrak{F}$ as the adjunction of a transcendental element x to the field $\mathfrak{F}$.

3.2. **The Hensel-Kürschák Field** $\mathfrak{X}_\xi$.—Let ξ be a real number $0 < \xi < 1$. We now define the modulus for the field $\mathfrak{X}_\xi$:

D 3.5 $$\| z^{\mathfrak{X}_\mathfrak{F}} \| \equiv 0; \quad \varphi \neq z^{\mathfrak{X}_\mathfrak{F}} \cdot) \cdot \| \varphi \| \equiv \xi^{\nu\varphi}.$$

The modular properties 1 and 4 of *D* 1.6 are obvious. The property 2 is also obvious in case one of the factors is the zero function. In the other cases,

$$\nu(\varphi_1\varphi_2) = \nu(\varphi_1) + \nu(\varphi_2) \quad (\varphi_1, \varphi_2).$$

Hence

$$\| \varphi_1\varphi_2 \| = \xi^{\nu(\varphi_1\varphi_2)} = \xi^{\nu(\varphi_1) + \nu(\varphi_2)} = \xi^{\nu(\varphi_1)}\xi^{\nu(\varphi_2)} = \| \varphi_1 \| \, \| \varphi_2 \|.$$

Thus 2 is proved completely.

Again, for every φ_1 and φ_2, $\nu(\varphi_1 + \varphi_2) =$ the first i for which $\varphi_1(i) + \varphi_2(i)$ is not zero (if there is any such, otherwise $\varphi_1 + \varphi_2 = z$ and the property 3′ is obvious); thus

$$\nu(\varphi_1 + \varphi_2) \geqq \text{the smaller of } \nu(\varphi_1),\ \nu(\varphi_2).$$

In this case since $0 < \xi < 1$, we have:

$$\varphi_1 \cdot \varphi_2 :) : \| \varphi_1 + \varphi_2 \| = \xi^{\nu(\varphi_1 + \varphi_2)} \leqq gr(\xi^{\nu(\varphi_1)}, \xi^{\nu(\varphi_2)}) \therefore \leqq gr(\| \varphi_1 \|, \| \varphi_2 \|).$$

Thus $\mathfrak{X}_\xi$ is a Hensel-Kürschák field (cf. *D* 2.4), denoting by $\mathfrak{X}_\xi$ the field $\mathfrak{X}_\mathfrak{F}$ with the modulus as defined in *D* 3.5.

Thm 3.2 $$\mathfrak{F} \cdot) \cdot \mathfrak{X}_\xi^{H}.$$

Hereafter we shall generally denote the elements of $\mathfrak{X}_\xi$ by $h(h_1, h_2, \text{etc.})$. The subscript $\mathfrak{F}$ in $\mathfrak{X}_\xi$ shall be dropped unless it is necessary to bring in evidence the field from which $\mathfrak{X}_\xi$ is formed. The elements of $\mathfrak{F}$ in $\mathfrak{X}_\xi$ (by isomorphism) will, when it is desirable to bring that fact in prominence, be denoted by $f(f_1, f_2, \text{etc.})$.

3.3. **Perfection of** $\mathfrak{X}_\xi$.—We now prove the theorem:

Thm 3.3 $$\mathfrak{F} \cdot) \cdot \mathfrak{X}_\xi^{P}.$$

Proof: Consider a Cauchy sequence $\{h_n\}$. From property 3′ of the modulus either $L_n \| h_n \| = 0$ in which case the limit of the sequence is z or

$$\exists\, (n_0, \nu) \ni \| h_n \| = \xi^\nu = \underset{n}{L} \| h_n \| \qquad (n > n_0)$$

for otherwise if $e < \underset{n}{L} \| h_n \|$ which exists, and $a \equiv \underset{n}{L} \| h_n \| - e$,

$$n > n_e :) : \exists\, (n_1, n_2) \ni \| h_{n_1} \| \neq \| h_{n_2} \| \cdot) \cdot \| h_{n_1} - h_{n_2} \|$$
$$= gr(\| h_{n_1} \|, \| h_{n_2} \|) > a$$

where n_e is the n_e for the original Cauchy sequence, and this is impossible. With this ν and $h_n \equiv \sum_i^{\nu-\infty} f_{n,-i}\, x^i\,(n)$, we have

$$\exists n_0 : \ni : n > n_0 \cdot) \cdot f_{n,\, i} = f_{n_0,\, i} \qquad (1)$$

for otherwise $n :) : \exists (n_1, n_2) > n \ni \| h_{n_1} - h_{n_2} \| \geqq \xi^{\nu-1}$, which is not true as $\{h_n\}$ is a Cauchy sequence. Similarly,

$$i :) : \exists n_{0,\, i} : \ni : n > n_{0,\, i} \cdot) \cdot f_{n,\, \nu+i} = f_{n_{0,\, i},\, \nu+i}. \qquad (2)$$

If we now define $h \equiv \sum_i^{\nu\infty} f_i x^i$, where $f_i \equiv f_{n_{0,\, i-\nu},\, i}$ $(i \geqq \nu)$ in (1) and (2), we see that $i :) : \| h - h_{n_{0,\, i}} \| \leqq \xi^{\nu+i}$ and so $h = \underset{n}{L} h_n$.

3.4. **Conditions for Compactness of $\mathfrak{X}_{\mathfrak{z}}$.**—Though every $\mathfrak{X}$ as we have seen is perfect, it is not necessarily compact. The necessary and sufficient conditions for compactness are given by the following theorem:

Thm 3.4 $\qquad \mathfrak{X}_{\mathfrak{z}}^{\text{cpt}} \cdot \sim \cdot \mathfrak{F}^{\text{cpt}} \cdot \sim \cdot \mathfrak{F}^{\text{finite}}.$

Proof: Since $f \neq z \cdot) \cdot \| f \| = 1$, $\underset{n}{L} f_n = h \cdot) \cdot h^{\mathfrak{z}}$; therefore if we have a Cauchy sequence of elements of $\mathfrak{F}$ its limit element, when it exists, must be an element of $\mathfrak{F}$. Take now an infinite sequence of elements of $\mathfrak{F}$. 1 is an upper bound of the modulus; therefore by $\mathfrak{X}_{\mathfrak{z}}^{\text{cpt}}$ this sequence has a Cauchy subsequence with a limit element. This limit element by the above is in $\mathfrak{F}$. Hence,

$$\mathfrak{X}_{\mathfrak{z}}^{\text{cpt}} .\,) \,.\, \mathfrak{F}^{\text{cpt}}.$$

It is also obvious that $\mathfrak{F}^{\text{finite}} \cdot) \cdot \mathfrak{F}^{\text{cpt}}$, for in this case every infinite sequence of elements of $\mathfrak{F}$ must have at least one of its elements repeated an infinite number of times.

Next suppose $\mathfrak{F}^{\text{cpt}}$; consider $\{h_n\} \ni \underset{n}{\overline{B}} \| h_n \| < \infty$, say $< \xi^t$. Let $h_n \equiv \sum f_{n,\, i}\, x^i\,(n)$. Then $f_{n,\, i} = z\,(n,\ i < t)$. Since every sequence in $\mathfrak{F}$ has 1 as an upper bound of the modulus, and hence by $\mathfrak{F}^{\text{cpt}}$ has a Cauchy subsequence with a limit element the sequence $\{f_{n,\, t}\}$ has a Cauchy subsequence with a limit element. Take the corresponding subsequence of $\{h_n\}$. If the coefficients in this sequence be denoted by letters with the upper subscript 1, we have a sequence whose terms begin with a power not less than t and the coefficients of x^t form a Cauchy sequence. Also since every non-zero element of $\mathfrak{F}$ has the modulus 1, after a finite number of terms, the terms in this coefficient sequence are identical. The coefficient sequence $\{f^{(1)}_{n,\, t+1}\}$ has similarly a Cauchy subsequence. Take the corresponding subsequence of $\{h_n\}$. If we continue in this manner, and denote the successive *distinct* subsequences of $\{h_n\}$ obtained in this manner by $\{h_n^{(1)}\}, \{h_n^{(2)}\}, \cdots, \{h_n^{(r)}\}, \cdots$, we have

$$\{h_n\} \supset \{h_n^{(r)}\} \supset \{h_n^{(r+1)}\} \quad , (r > 0) \qquad (1)$$

and

$$r > 0 :) : \exists n_r : \ni : m > r \cdot n > n_r \cdot i < t + r :) : f^{(m)}_{n,\ i} = f_{n_r,\ i}. \qquad (2)$$

If this series of subsequences has a last term, say $\{h^{(r)}_n\}$, then its coefficient sequences are all Cauchy sequences and it can be easily proved that $\{h^{(r)}_n\}$ is itself a Cauchy sequence, and therefore by $\mathfrak{X}^c_{\mathfrak{z}}$, with a limit element in $\mathfrak{X}_{\mathfrak{z}}$.

If the series of subsequences has no last term, take the subsequence of $\{h_n\}$ formed in the following manner: from each term of the infinite series select the first element which does not belong to the next term in the sequence (such elements do exist, since the terms of the infinite series are distinct). By using (2), this sequence can be seen to be a Cauchy sequence and therefore by the perfection of $\mathfrak{X}_{\mathfrak{z}}$, with a limit element in $\mathfrak{X}_{\mathfrak{z}}$.

Finally, to prove $\mathfrak{F}^{\text{cpt}} \cdot) \cdot \mathfrak{F}^{\text{finite}}$: We prove this by proving directly the contrapositive, viz.,

$$\mathfrak{F}^{\text{infinite}} \cdot) \cdot \mathfrak{F}^{-\text{cpt}}.$$

For consider $\{f_n\}^{\text{distinct}}$; then $\overline{B}_n \| f_n \| = 1$. If this sequence has a Cauchy subsequence, $e \cdot) \cdot \exists (n_1, n_2) \ni \| f_{n_1} - f_{n_2} \| < e$; but by $\{f_n\}^{\text{distinct}}$ and the fact that $f \neq z \cdot) \cdot \| f \| = 1$, we have $n_1 \neq n_2 \cdot) \cdot \| f_{n_1} - f_{n_2} \| = 1$, and these contradict each other when $e < 1$. Hence the result.

Thus the theorem is completely proved.

Corollary: Since every finite field is of the type* $[p^q]$, where p is prime and q a positive integer, we have:

Thm 3.5 $$\mathfrak{X}^{\text{cpt}}_{\mathfrak{z}} \cdot \sim \cdot \mathfrak{F}^{\text{type } [p^q]}.$$

3.5. **Subfield** $R(\mathfrak{F}, x)$ **of** $\mathfrak{X}_{\mathfrak{z}}$.—Since rational integral functions of a transcendental x with coefficients in $\mathfrak{F}$ form a subclass of $\mathfrak{X}_{\mathfrak{z}}$, viz., the class of all finitely non-zero functions on $\mathfrak{F}$ to $\mathfrak{F}$, with z as the functional value for negative values of the argument, the field of all rational functions of such an x is a subfield of $\mathfrak{X}_{\mathfrak{z}}$. Thus,

Thm 3.6 $$\mathfrak{F} \cdot) \cdot \mathfrak{X}_{\mathfrak{z}} \supset R(\mathfrak{F}, x).$$

Further,†

Thm 3.7 $$\mathfrak{F}^{\text{finite or of the type } \mathfrak{S}^p} \cdot) \cdot \mathfrak{X}_{\mathfrak{z}} \underset{\neq}{\supset} R(\mathfrak{F}, x).$$

Proof: The proof is partly suggested by Hensel's proof of the theorem that the field of the p-adic numbers includes properly the field of ordinary rationals.‡ In analogy with Hensel we proceed to show that under the conditions in the hypotheses every element of $R(\mathfrak{F}, x)$ corresponds to a

* M, p. 220.

† Here the symbol $\underset{\neq}{\cup}$ reads "*properly includes.*"

‡ H, p. 38.

periodic element of $\mathfrak{X}_{\mathfrak{z}}$. A *periodic element* is defined thus:

D 3.6 $\quad (h \equiv \sum f_i x^i)^{\text{periodic}} : \equiv : h : \ni : \exists (s, t > 0) \ni i > s \cdot) \cdot f_i = f_{i+t};$

the smallest such t is called *the period.*

All polynomials are obviously periodic, the period being 1 and the repeated element z. The product of two periodic elements is easily seen to be periodic. Thus it is sufficient to prove that the reciprocal of a polynomial φ irreducible in $\mathfrak{F}$ is a periodic element.

In case $\mathfrak{F}$ is of the type $\mathfrak{P}^p$, that is, the absolute algebraic field * characteristic p, φ is of the first degree in x. In case $\varphi = fx$, f being a non-zero element of $\mathfrak{F}$, the reciprocal of φ is $f'x^{-1}$, where f' is the reciprocal of f, and is, therefore, obviously periodic. In case $\varphi = fx + f_1$ where none of f, f_1 are zero, its reciprocal is $f_1'/(1 + f_2 x)$, where f_1' is the reciprocal of f_1 and $f = f_1 f_2$. But $1/(1 + f_2 x)$ equals:

$$1 - f_2 x + f_2^2 x^2 - \cdots + (-1)^r f_2^r x^r + \cdots$$

and since f_2 is algebraic with respect to the field $[0, 1, 2, \cdots, p-1]$ there exists† a positive integer n such that $f_2^n = 1$. Hence the series is periodic.

In case $\mathfrak{F}$ is finite, $\mathfrak{F}$ is a Galois Field ‡ $[p^q]$. If the polynomial φ irreducible in $\mathfrak{F}$ is of degree n in x, the class of all polynomials ψ in x with coefficients in $\mathfrak{F}$ falls modulo φ into p^{nq} classes of congruent polynomials, and hence there exists § a positive integer m such that $x^m \equiv 1 \pmod{\varphi}$, that is to say, there exists a polynomial ψ such that $\varphi(x)\psi(x) = x^m - 1$. Thus:

$$1/\varphi(x) = \psi(x)/[\varphi(x)\psi(x)] = \psi(x)/(x^m - 1) - - \psi(x)\{1 + x^m + x^{2m} + \cdots\}$$

and is therefore obviously periodic.

3.6. **Adjunction of Elements to $\mathfrak{X}_{\mathfrak{z}}$ algebraic to $\mathfrak{F}$.**—We shall now prove a theorem required for the sequel:

Thm 3.8 $\quad \mathfrak{F} \cdot S^{\text{alg } \mathfrak{z} \cdot}) \cdot \mathfrak{X}_{\mathfrak{z}(S)} = \mathfrak{X}_{\mathfrak{z}}(S),$

where S denotes a system of elements.

Proof: We put the elements of the two fields into isomorphism in this manner: put the systems $\mathfrak{F}$ and S in one into correspondence with those in the other; since S is algebraic with respect to $\mathfrak{F}$, this correspondence will persist under all the field operations like additions, etc. Also since the moduli of these elements in $\mathfrak{F}$ and S are the same in both cases the iso-

* Cf. 1.2; also S, p. 199.

† S., p. 251.

‡ M, p. 220.

§ This can be proved by the usual methods of Galois Field theory (cf. L. E. Dickson, "History of the Theory of Numbers," Washington, D. C., 1919).

morphism so far is complete. Also both $\mathfrak{X}_{\mathfrak{z}(S)}$ and $\mathfrak{X}_{\mathfrak{z}}(S)$ are extensions of $\mathfrak{F}(S)$. If we now identify the element x and its moduli in the two fields, we see that both $\mathfrak{X}_{\mathfrak{z}(S)}$ and $\mathfrak{X}_{\mathfrak{z}}(S)$ are extensions of $\mathfrak{X}_{\mathfrak{z}}$. Now $\mathfrak{X}_{\mathfrak{z}}(S)$ is the smallest extension* of $\mathfrak{X}_{\mathfrak{z}}$ containing the system S. Therefore $\mathfrak{X}_{\mathfrak{z}(S)} \supset \mathfrak{X}_{\mathfrak{z}}(S)$. Further every element of $\mathfrak{X}_{\mathfrak{z}(S)}$ is of the form $\sum_i s_i x^i$, where the elements s belong to $\mathfrak{F}(S)$, and is thus an element of $\mathfrak{X}_{\mathfrak{z}}(S)$. Thus the theorem is completely proved.

IV. THE PROPERTIES p, A, P, *cpt*.

4.1. **A Complete Existential Theory.**—In the preceding sections we have defined, with reference to Kürschák fields, the four properties p, A, P, and *cpt*. If the existence or non-existence of these properties in this order is denoted by positive or negative signs in that order, a given Kürschák field will have one of the $2^4 = 16$ characters $(++++)$, $(+++-)$, $(++-+)$, $\cdots$, $(----)$. In this section we develop a *complete existential theory*† of these properties, that is, we prove 2^4 propositions stating for each of the 2^4 combinations $(++++)$, $\cdots$, $(----)$ of these properties whether there exists or does not exist a Kürschák field having the particular combination of the properties. These propositions are tabulated below:

4.1–4.16:

Cases.	p.	A.	P.	*cpt*.	Consistent.
1	+	+	+	+	−
2	+	+	+	−	+
3	+	+	−	+	−
4	+	+	−	−	+
5	+	−	+	+	+
6	+	−	+	−	+
7	+	−	−	+	−
8	+	−	−	−	+
9	−	+	+	+	+
10	−	+	+	−	+
11	−	+	−	+	−
12	−	+	−	−	+
13	−	−	+	+	+
14	−	−	+	−	+
15	−	−	−	+	−
16	−	−	−	−	+

Note: The + and − entries in the last column denote the existence and non-existence respectively of a Kürschák field with the character denoted by the entries under the first four columns.

Also in looking the entries under p, it is necessary to remember that the entries in the final column do not depend upon any special choice of the

* S, p. 186.

† E. H. Moore, "Introduction to a Form of General Analysis," 1910, Yale University Press, p. 82.

as the entries with + sign under p we interpret as $p\cdot) \cdot {}^{\pm}\exists$, and those with − sign as ${}^{\pm}\exists\ \mathfrak{R}^{0.\ \text{etc.}}$.

4.2. *Proofs:*

(*a*) 3, 7, 11, and 15 follow from theorem 2.1, viz., $\mathfrak{R}^{\text{cpt}}\cdot)\cdot\mathfrak{R}^{P}$.

(*b*) 1 follows from theorem 2.4, viz., $\mathfrak{R}^{A.\ \text{cpt}}\cdot)\cdot\mathfrak{R}^{0}$.

(*c*) The following number systems in analysis with the absolute value as modulus prove 9, 12, 13, and 16 respectively:

(9) complex numbers,
(12) algebraic numbers,
(13) real numbers,
(16) rational numbers.

(*d*) To prove 2: consider $(\mathfrak{X}_{\mathfrak{p}^p})_{(AP)}$; this has the properties p, A, P. By theorem 3.4, $\mathfrak{X}_{\mathfrak{p}^p}$ is P and $-$ *cpt*. Using 2.15 we see that $(\mathfrak{X}_{\mathfrak{p}^p})_{(AP)}$ is $-$ *cpt*.

(*e*) To prove 5: consider $\mathfrak{X}_{[p]}$. This field is ^{-}A, for the equation $y^2 - x = 0$ has no solution. However by 3.4 it is *cpt*.

(*f*) We get 6, an example being $\mathfrak{X}_{\mathfrak{p}^p}$. The proof is similar to the above.

(*g*) Similarly $\mathfrak{X}_{\mathrm{R}}$ where $\mathrm{R} \equiv$ [all ordinary rationals] gives us 14.

4.3. *Proof of 10:* We proceed to prove that:

Thm 4.10$_0$ $\qquad \mathfrak{S}^{-\text{cpt}}_{(AP)}$,

where $\mathfrak{S}$ is the field of Hensel's $\mathring{p}$-adic numbers.

Take an element of $\mathfrak{S}$ whose modulus a is > 1. Take an infinite sequence of positive distinct ordinary rationals $\{r_n\}$ so that they all lie between two positive rationals r, r'. Let $r_n \equiv l_n/m_n$, where l_n, m_n are positive and relatively prime. Consider the sequence $\{h_n\}$ where $h_n^{m_n} \quad h_n^{l_n} = 0\ (n)$. Then $\|h_n\| = a^{r_n}\ (n)\quad \therefore \{h_n\}^{\text{distinct}}$. This sequence an upper bound of whose modulus is $gr(a^r, a^{r'})$, cannot have any Cauchy subsequence.

For otherwise,

$$e:):\exists\, n_e \ni (n_1, n_2) > n_e\cdot)\cdot \|h_{n_1} - h_{n_2}\| < e,$$

but we have

$$n_1 \neq n_2 :):\|h_{n_1}\| \neq \|h_{n_2}\| :):\|h_{n_1} - h_{n_2}\| = gr(\|h_{n_1}\|, \|h_{n_2}\|) \quad \therefore \geq a^r,$$

and taking $e < a^r$, we get a contradiction.

Hence the theorem.

4.4. *Proof of* 4: Consider $(\mathfrak{X}_{[p]})_A$; $\mathfrak{X}_{[p]}$ is perfect by 3.3. If we now prove that algebraic closure is in this case obtained by the adjunction of an infinite number of elements, then, by 2.21 $(\mathfrak{X}_{[p]})^{-P}_A$ and hence by 2.1

— *cpt.* Now $(\mathfrak{X}_{[p]})_A$ contains $\mathfrak{X}_{\mathfrak{P}^p}$ which by theorem 3.8 can be as obtained from $\mathfrak{X}_{[p]}$ by adjoining algebraic elements to $[p]$. The number of these is, however, infinite; otherwise $\mathfrak{P}^p$ would have only a finite number of elements. Thus $(\mathfrak{X}_{[p]})_A$ is infinite with respect to $\mathfrak{X}_{[p]}$ and so the theorem is proved.

4.5. *Proof of 8:* Ostrowski in his theses* has proved the following theorem:

Thm 4.8$_0$ $\qquad \mathfrak{K}^P \cdot S^{\text{algi } \mathfrak{K}; \text{ infinite}} \cdot) \cdot \mathfrak{K}(S)^{-P}.$

We use this theorem to prove 4.8.

Consider $\mathfrak{X}_{[p]}$, $p > 2$; let r be any one of the integers: $2, 3, \cdots, p-1$. From Steinitz's paper,† we have

$$q^{\text{prime.} > p \cdot} :) : (y^q + r)^{\text{irreducible in } [p]. \text{ Ist kind}}.$$

Consider a sequence of primes, $\{q_n\}$ such that

$$(1)\ q_1 > p$$
$$(2)\ q_n > p^{q_1 q_2 q_3 \cdots q_{n-1}}$$

and the corresponding infinite system of irreducible polynomials

$$S \equiv (y^{q_n} + r \mid n).$$

Consider now $\mathfrak{X}_{[p]}(S)$. By theorem 3.8, $\mathfrak{X}_{[p]}(S) = \mathfrak{X}_{[p](S)}$. Also S is of the first kind, further if $S_m \equiv (q^{q_n} + r \mid n = 1, 2, \cdots, m-1)$, $[p](S_m)$ contains elements whose order is at most $p^{q_1 q_2 \cdots q_{m-1}}$ and hence the polynomial $y^{q_m} + r$ is irreducible in $[p](S_m)$ since the order of its roots is $q_m >$ the highest order in $[p](S_m)$. Thus S is a progressive ‡ system. We have further $\mathfrak{X}_{[p]}(S)^{-A}$, since the polynomial $y^2 + x$ for instance has still no root. Thus $\mathfrak{X}_{[p]}(S)^{p-A-P-\text{cpt}}$.

* O_1, p. 280.

† S, p. 231.

‡ S, p. 271.